Orchard Units

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One Page Summaries

Euclid's Orchard

freeman66 and naman12

Contents

I

Prologue

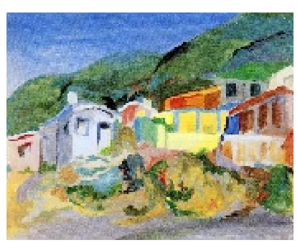
1 Acknowledgements

This was made for the Art of Problem Solving Community out there! We would like to thank Evan Chen for his evan.sty code. In addition, all problems in the handout were either copied from the Art of Problem Solving Wiki or made by ourselves.

Art of Problem Solving Community

Evan Chen's Personal Sty File

NAMAN12's Website: Say hi!



FREEMAN66's Website: Say hi!

Note: This is a painting by Richard Feynman, who I admire a lot.

And Evan says he would like this here for evan.sty:

2 Introduction

Contents

2.1 Motivation and Goals

Most people are too lazy to read handouts. The substitute to this is to attend classes and learn face to face how certain techniques work. This book is for those who have already done that, and want a refresher on what they learned. The goal is that this book can serve as quick list of what to study, as well as the key points of each topic.

2.1.1 A Small Warning

Do not use this this wrong! In other words, don't expect too much detail, and don't expect any examples to try. This is not a handout, but rather a formulary. It outlines important topics, but doesn't go into depth over them. I recommend searching the things you don't know, and contacting us if you want more information on specific topics. See the Contact section below for more information on guidelines of contacting us.

2.2 Contact

Do you have questions, comments, concerns, issues, or suggestions? Here are two ways to contact freeman66 or naman12:

1. Send an email to realnaman12@gmail.com or dylanyu66@gmail.com. This is the best method.

2. Send a private message to naman12 or freeman66 by either clicking the button that says PM or by going here and clicking New Message and typing naman12 or freeman66.

Please include something related to **One Page Summaries** in the subject line so freeman66 or naman12 knows what you are talking about. Our website can be found here. Note that if you have **any** suggestion on a summary, please let us know! We would love to hear it. For example, if you think a specific section of a chapter should be expanded on, let us know! We can write a one page summary for that specific topic.

II

Algebra

Contents

3 Operations

3.1 Definitions

3.1.1 Basics

Definition 3.1.1 (Operation) — An **operation** is a function which takes input values (called **operands**) to a output value.

Definition 3.1.2 (Arity) — The number of operands is the **arity** of the operation.

3.1.2 Arity

- A **nullary** operation has arity 0. Ex: constant.
- A **unary** operation has arity 1. Ex: additive inverse, multiplicative inverse.
- A **binary** operation has arity 2. Ex: addition, multiplication, subtraction, division.
- A **finitary** operation has a finite arity,
- An **infinitary** operation has an infinite arity.

3.2 The Classic Four

Definition 3.2.1 (Addition) — The **addition** of two numbers is the total amount of those values combined.

Definition 3.2.2 (Subtraction) — **Subtraction** is an arithmetic operation that represents the operation of removing objects from a collection.

Definition 3.2.3 (Multiplication) — **Multiplication** is repeated addition.

Definition 3.2.4 (Division) — **Division** is the way that numbers are combined to make new numbers.

Here is a bit more vocabulary:

1. **Addend:** The numbers used to make the addition.
2. **Difference:** The result of a subtraction.
3. **Multiplicand:** What is being multiplied.
4. **Multiplier:** How much the multiplicand is being multiplied by.
5. **Dividend:** What is being divided.
6. **Divisor:** What the dividend is being divided by.
7. **Quotient:** The result of a division.
8. **Remainder:** What is left over, or what cannot be fully divided.

3.3 Other Operations

Definition 3.3.1 (Exponentiation) — **Exponentiation** is repeated multiplication. This is usually written as a^b.

Definition 3.3.2 (Logarithm) — The **logarithm** of a number is how many times repeated multiplication can be done. This is usually written as $\log_a b$. The variable b is known as the **base**.

Definition 3.3.3 (nth Root) — The **nth root** of a number is the number that is multiplied by itself n times to equal a given value. It is commonly written as $\sqrt[n]{a}$.

3.3.1 The Power of Operations

What make operations useful are its conciseness. If we had to do $2+2+2+2+2+2+2$ every time we wanted to write $7 \cdot 2$, math wouldn't progress nearly as fast as it does now. For example, the Ackermann's function rises so fast that after only a few iterations the number is too big to comprehend (literally! Graham's number is more than the number of atom's in the visible universe).

3.3.2 Operation Problems

Sometimes, a problem will ask for $5 \star 3$ if $a \star b = a^2 - 2ab + b^2$. This requires algebraic manipulation, which will be discussed in Variables.

4 Variables

4.1 Definitions

Definition 4.1.1 (Variable) — A **variable** is something that can change. It can also be a symbol for a number we don't know yet.

There are **independent** and **dependent** variables. *Independent* variables do not affect one another.

Definition 4.1.2 (Ratio) — A **ratio** shows the relative sizes of two or more values. The : symbol represents the word "to".

Definition 4.1.3 (Proportion) — A **proportion** is a statement that two *ratios* are equal.

Two numbers x, y are said to be *directly proportional* if

$$y = kx$$

for some constant k. Two numbers a, b are said to be *inversely proportional* if

$$ab = k$$

for some constant k. *Joint proportions* occur when $x = kyz$, for some constant k.

4.2 d = rt

The infamous equation:

Theorem 4.2.1 ($d = rt$) — Like the title of this theorem says,

$$\text{distance} = \text{rate} \times \text{time}.$$

Theorem 4.2.2 (Working Together) — Let person 1 work at a speed of one object per t_1 time, person 2 works at a speed of one object per t_2 time, and so on, all the way to person n. Working together, they will finish in

$$\frac{1}{\frac{1}{t_1} + \frac{1}{t_2} + \ldots + \frac{1}{t_n}} \text{ time.}$$

Theorem 4.2.3 (Moles Digging Holes) — If we have a moles, b holes, and c hours, then a varies directly with b, b varies directly with c, and c varies inversely with a.

4.3 Simon's Favorite Factoring Trick

Theorem 4.3.1 (SFFT) — For all real numbers (although commonly used only for integers) x, y, a, b,

$$xy + xa + yb + ab = (x + b)(y + a).$$

4.4 Algebraic Manipulation

Theorem 4.4.1 (Egyptian Fractions) — For all a, b where $ab \neq 1$,

$$\frac{a}{ab - 1} = \frac{1}{b(ab - 1)} + \frac{1}{b}.$$

4.4.1 Memorization

Theorem 4.4.2 (Common Manipulations) — Let x, y be nonzero real numbers such that $x + y = a$ and $xy = b$. Then,

$$x^2 + y^2 = a^2 - 2b,$$

$$(x + 1)(y + 1) = a + b + 1,$$
$$x^2y + xy^2 = ab,$$
$$x^3 + y^3 = a^3 - 3ab,$$
$$\frac{1}{x} + \frac{1}{y} = \frac{a}{b}.$$

A few nice formulas:

- $a^2 - b^2 = (a + b)(a - b)$
- $(a \pm b)^2 = a^2 \pm 2ab + b^2$
- $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$
- $a^3 \pm b^3 = (a + b)(a^2 \mp ab + b^2)$
- $(a + b \pm c)^2 = a^2 + b^2 + c^2 - 2(\mp ab + \mp bc - ca)$
- $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$
- $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab+bc+ca}{abc}$

5 Linear & Quadratic Equations

5.1 Definitions

Definition 5.1.1 (Linear) — A **linear** equation is a polynomial with an x term and no higher terms.

Definition 5.1.2 (Quadratic) — A **quadratic** equation is a polynomial with an x^2 term and no higher terms.

5.2 Linear Equations

5.2.1 Form of a Linear Equation

1. Standard Form: $ax + by + c = 0$, or sometimes $ax + by = c$, where a, b, c are constants, and $-\frac{a}{b}$ is the slope.

2. Slope-Intercept Form: $y = mx + b$, where m is the slope and b is the y-intercept, which is where the y-axis intersects the line. Similarly the x-intercept is the point where the x-axis intersects the line.

3. Point-Slope Form: $y - y_0 = m(x - x_0)$, where (x_0, y_0) is a point on the line, and m is the slope of the line.

5.2.2 Solving Linear Equations

Theorem 5.2.1 (Substitution Method) — Solve for one of the variables, and plug that in.

Theorem 5.2.2 (Elimination Method) — Subtracting a variable with the same coefficient with cancel it out.

5.3 Quadratic Equations

5.3.1 Solving Quadratics

There are quite a few ways to solve quadratics, including: factoring, completing the square, and the quadratic formula. Factoring can be seen here and here.

Theorem 5.3.1 (Completing the Square) — Let $f(x) = ax^2 + bx + c$. Then f can be written as

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c.$$

While this doesn't look very useful, if the expression is nice, this can help reduce computation. If we directly solve for x, we get:

Theorem 5.3.2 (Quadratic Formula) — Let $f(x) = ax^2 + bx + c$. Then if $f(x_1) = f(x_2) = 0$,

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Po Shen Loh recently (as of 2020) discovered a method of solving quadratics, located here.

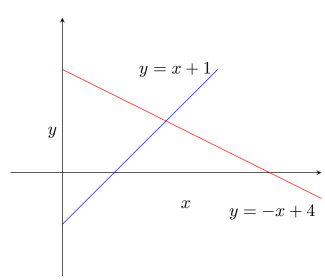
5.3.2 Discriminant

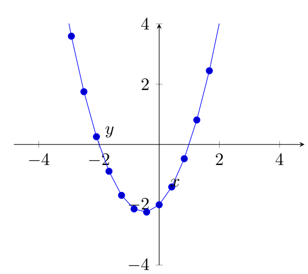
To tell how many real roots there are, look at the discriminant. In general, if $b^2 - 4ac$ is:

1. **greater than 0**: there are 2 real roots.
2. **equal to 0**: there is 1 real root.
3. **less than 0**: both roots are complex.

5.4 Graphing

The graph on the left illustrates what linear equations look like: lines. Find two points, and you can draw a line! To graph a **parabola** (right graph), you must plot enough points to get the basic shape.





The lowest/highest point on a parabola is known as the **vertex**. It always has an x-value of $-\frac{b}{2a}$. A parabola opens **up** if $a > 0$, and opens **down** if $a < 0$.

6 Polynomials

6.1 Definitions

Definition 6.1.1 (Polynomial) — A **polynomial** is a function that consists of a sum of variables raised to powers and multiplied by coefficients.

Definition 6.1.2 (Degree) — The **degree** of a polynomial is its highest (sum of) exponent in a single term.

Definition 6.1.3 (Root) — A **root** of a polynomial $P(x)$ is a value r such that $P(r) = 0$.

6.2 Fundamentals

Theorem 6.2.1 (Unique Factorization) — Any polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ can be expressed as $f(x) = a_n(x - r_1)(x - r_2)\ldots(x - r_n)$, where $r_1, r_2, \ldots, r_n$ are the roots of $f(x)$.

Theorem 6.2.2 (Fundamental Theorem of Algebra) — Given a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, there exists a root $r \in \mathbb{C}$.

Theorem 6.2.3 (Factor Theorem) — $x - r$ is a factor of $f(x)$ if and only if $f(r) = 0$.

Theorem 6.2.4 (Remainder Theorem) — $f(k)$ is the remainder when $f(x)$ is divided by $x - k$.

6.3 Roots

Theorem 6.3.1 (Rational Root Theorem) — Given a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ with integral coefficients, $a_n \neq 0$. The Rational Root Theorem states that if $P(x)$ has a rational root $r = \pm\frac{p}{q}$ with p, q relatively prime positive integers, p is a divisor of a_0 and q is a divisor of a_n.

Theorem 6.3.2 (Vieta's Formulas) — Suppose that the roots to $f(x) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_0$ are $r_1, r_2, \ldots, r_n$. Then, we get that if σ_k is the symmetric sum taking the r_i k at a time, then

$$\sigma_k = (-1)^k \frac{c_{n-k}}{c_n}$$

6.4 Polynomial Techniques

Theorem 6.4.1 (Newton's Formulas) — Let ρ_k be $x_1^k + x_2^k + \ldots + x_n^k$. Then, we get

$$k\sigma_k + \sum_{j=1}^{k} (-1)^j \sigma_{k-j}\rho_j = 0,$$

where we define for $j > n$ and $j < 0$ $\sigma_j = 0$ and for $j = 0$ $\sigma_j = 1$.

Theorem 6.4.2 (Difference of Polynomials) — Let $P(x)$ be a polynomial with integer coefficients. Then, we have that $a - b \mid P(a) - P(b)$.

6.5 Manipulations for Polynomials

Here is a short list, where $u = \sigma_1, v = \sigma_2, w = \sigma_3$:

1. $a^2 + b^2 + c^2 = u^2 - 2v$
2. $a^2b^2 + b^2c^2 + c^2a^2 = v^2 - 2uw$
3. $(a + b)(b + c)(c + a) = uv - w$
4. $(1 + a)(1 + b)(1 + c) = 1 + u + v + w$

Theorem 6.5.1 (Reversed Polynomial Coefficients) — Let a_i be the coefficient of the x^i term in $P(x)$. Then the coefficient of x^i in $x^n P\left(\frac{1}{x}\right)$ is a_{n-i}.

7 Functions

7.1 Definitions

Definition 7.1.1 (Function) — A **function** is a rule that maps one set of values to another set of values.

A rigourous definition is given here. Functions are usually written like $f(x)$ or f, and if we had $f(x) = x + 1$, then $f(1) = 1 + 1 = 2$, $f(2) = 3, \ldots$. The **domain** of a function is the set of input values for the argument of a function. The **range** of a function is the set of output values for that function.

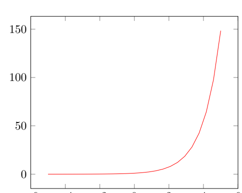
Definition 7.1.2 (Inverse) — The **inverse** f^{-1} of a function f has the property that $f(f^{-1}(x)) = x$ (called the **right** inverse), and $f^{-1}(f(x)) = x$ (called the **left** inverse).

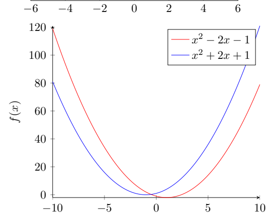
Definition 7.1.3 (Monotonic Function) — A function is **monotonically increasing** if $f(a) \geq f(b)$ for all $a > b$. It is **monotonically decreasing** if $f(a) \leq f(b)$ for all $a < b$.

A function is **continuous** if its graph can be drawn without taking the pencil off the paper. Note that this is not rigorous, but good enough for most cases. If you would like, take a look at the Epsilon-Delta definition or the Heine definition.

7.2 Graphing

There are very powerful graphing calculators nowadays, including Desmos. You can learn more about graphing here or here. Khan Academy has many videos on this as well. The following are examples of 2D and 3D functions.





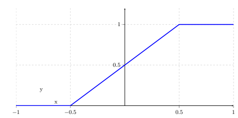
7.3 Types of Functions

7.3.1 Recursive Functions

A **recursive function** is defined upon previous values of the function. For example, $f(n) = f(n-1) + f(n-2)$ is recursive. Recursive functions are usually rational or integral, and appear a lot in AMC and AIME questions.

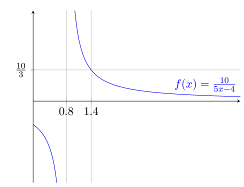
7.3.2 Piecewise Functions

Piecewise functions are parts of various functions smushed together into one function. See the right figure.



7.3.3 Rational Functions

A **rational function** is a function that is the ratio of two polynomials. See the right figure.



7.4 Functional Equations

A **functional equation** is any equation in which the unknown represents a function. Functional equations can be solved by bounding and sometimes even plugging in nice values. More information can be found here.

7.4.1 Types of Functional Equations

An **injection** (or one-to-one function) is a function which always gives distinct values for distinct arguments. A **surjection** (or onto function) maps at least one element from its domain onto every element of its range. A **bijection** (or one-to-one correspondence, which must be one-to-one and onto) is a function that is both injective and surjective.

7.4.2 Common Functional Equations

1. **Cauchy**: Let $f(x+y) = f(x) + f(y)$. Then if $f : \mathbb{Q} \to \mathbb{Q}$, f is linear. The $\mathbb{R}$ case is similar, but with more conditions on f.

2. **Jensen**: Let $f(x) + f(y) = 2f\left(\frac{x+y}{2}\right)$. Then if $f : \mathbb{Q} \to \mathbb{Q}$, f is linear.

More can be found here.

8 Sequences & Series

8.1 Definitions

Definition 8.1.1 (Sequence) — A **sequence** is an ordered list of numbers.

Definition 8.1.2 (Partial Sum) — A **partial sum** is the sum of a portion of the sequence.

Definition 8.1.3 (Series) — A **series** is the sum of the elements in a given sequence.

An **arithmetic sequence** is a sequence of numbers in which consecutive terms are the same distance apart. A **geometric sequence** is a sequence of numbers in which each term is a fixed multiple of the previous. An **infinite** geometric sequence is a geometric sequence with an infinite number of terms, whereas a **finite** geometric sequence only has a finite number of terms.

8.1.1 Summation Notation

Summation means that for x in range $a \leq x \leq b$ that all values x will be tested in function $f(x)$ and summed together. The general notation from our example would be

$$\sum_{x=a}^{b} f(x) = f(a) + f(a+1) + \ldots + f(b-1) + f(b).$$

8.1.2 Arithmetic Sequences

Theorem 8.1.4 (Arithmetic Sequence Terms) — The n^{th} term in an arithmetic sequence is described $a_n = a_1 + d(n-1)$, where a_n is the n^{th} term, a_1 is the first term, and d is the difference between consecutive terms.

Theorem 8.1.5 (Arithmetic Sequence Series) — The sum of the first n terms of an arithmetic sequences is

$$s_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n-1)d).$$

From this, we can derive the sum of the first n integers, n odd integers, and n even integers.

8.1.3 Geometric Sequences

A geometric sequence may be defined recursively by $a_n = r \cdot a_{n-1}, n > 1$, with a fixed first term a_1 and common ratio r. Using this definition, the nth term has the closed-form: $a_n = a_1 \cdot r^{n-1}$.

Theorem 8.1.6 (Infinite Geometric Series) — Let a be the first term and let r be the common ratio. Then

$$S_\infty = \frac{a}{1-r}.$$

Theorem 8.1.7 (Finite Geometric Series) — Let a be the first term, let r be the common ration, and let n be the number of terms. Then

$$S_n = \frac{a(1-r^{n+1})}{1-r}.$$

8.1.4 Arithmetico-Geometric Sequences

An **arithmetico-geometric sequence** is a sequence of the form $t_n = a_n g_n$, where a_n and g_n are the nth terms of arithmetic and geometric sequences, respectively.

Theorem 8.1.8 (Finite Arithmetico-Geometric Series) — The sum of the first n terms of an arithmetico-geometric sequence is

$$\frac{a_n g_{n+1}}{r-1} - \frac{x_1}{r-1} - \frac{d(g_{n+1} - g_2)}{(r-1)^2},$$

where d is the common difference of a_n and r is the common ratio of g_n.

8.1.5 Telescoping Series

A **telescoping** series is a series whose partial sums have a fixed number of terms after cancellation. The idea is usually to use partial fraction decomposition.

8.1.6 Recursive Sequences

A **recursive sequence** is a sequence defined upon previous terms. Common examples include the Fibonacci and Catalan Sequences.

III

Appendix

Contents

Appendix A: List of Theorems and Definitions

A.1 List of Theorems

A.2 List of Definitions