Hidden Gems

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Here are some problems that don't get the love they deserve. Most of them were taken from David Altizio's collection.

1 Algebra

Problem 1 (ML HS 2001). What is the only ordered pair of real numbers (x, y) that satisfies

$$7^x - 11y = 0,$$
$$11^x - 7y = 0?$$

Problem 2 (Mandelbrot 2002). Let c be the smallest real solution to the equation

$$3^x = x + 2.$$

To six decimal places, $c = -1.87213$. Calculate the value of $3^{(3^c)}$, rounded to the nearest hundredth.

Problem 3 (Mandelbrot 2003). Supose that $f(x)$ and $g(x)$ are functions which satisfy $f(g(x)) = x^2$ and $g(f(x)) = x^3$ for all $x \geq 1$. If $g(16) = 16$, then compute $g(4)$.

Problem 4 (ML HS 1991). If x is not the square of an integer, then the value of

$$7 + \sqrt{x} + \frac{1}{5 - \sqrt{x}}$$

is a rational number for only one positive integer x. What is this value of x?

Problem 5. In 1734, Euler solved the Basel Problem, which involved proving that:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Now it's your turn. Compute the value of:

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots.$$

Problem 6 (HMMT February Guts 2018/8). Suppose a real number $x>1$ satisfies

$$\log_2(\log_4 x)+\log_4(\log_{16} x)+\log_{16}(\log_2 x)=0.$$

Compute

$$\log_2(\log_{16} x)+\log_{16}(\log_4 x)+\log_4(\log_2 x).$$

Problem 7 (Mandelbrot 2009). Find a quadratic $f(x)=x^2+ax+b$ such that

$$\frac{f(f(x)+x)}{f(x)}=x^2+1776x+2010.$$

Problem 8 (ARO 2001/10.5). The polynomial $P(x)=x^3+ax^2+bx+d$ has three distinct real roots. The polynomial $P(Q(x))$, where $Q(x)=x^2+x+2001$, has no real roots. Prove that $P(2001)>\frac{1}{64}$.

Problem 9 (HMMT February Algebra-NT 2017/1). Let $Q(x)=a_0+a_1x+\cdots+a_nx^n$ be a polynomial with integer coefficients, and $0\le a_i<3$ for all $0\le i\le n$.

Given that $Q(\sqrt{3})=20+17\sqrt{3}$, compute $Q(2)$.

2 Combinatorics

Problem 10. The numbers $1,2,\ldots,20$ are written on the board. One is allowed to erase any two numbers a and b and instead write the number $a+b-1$. What number can remain on the board after 19 such operations?

Problem 11 (HMMT February Guts 2017/9). Jeffrey writes the numbers 1 and 10^8 on the blackboard. Every minute, if x,y are on the board, Jeffery replaces them with

$$\frac{x+y}{2} \quad\text{and}\quad 2\left(\frac{1}{x}+\frac{1}{y}\right)^{-1}.$$

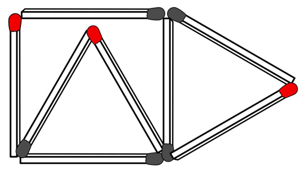
After 2017 minutes the two numbers are a and b. Find $\min(a,b)$ to the nearest integer.

Problem 12 (ARO 1998/9.8). Two distinct positive integers a,b are written on the board. The smaller of them is erased and replaced with the number $\frac{ab}{|a-b|}$. This process is repeated as long as the two numbers are not equal. Prove that eventually the two numbers on the board will be equal.

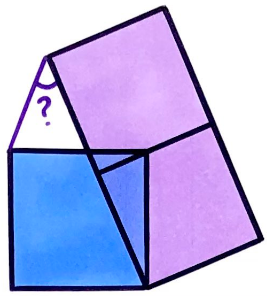
3 Geometry

Problem 13 (Mandelbrot 1999). We are given points A, B, C, and D in the plane such that $AD = 13$ while $AB = BC = AC = CD = 10$. Compute $\angle ADB$ in degrees.

Problem 14. Prove that the red ends of the matchsticks shown below are collinear.



Problem 15 (Catriona Shearer). Find the angle.



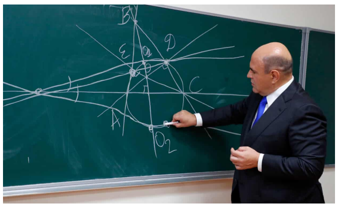
Problem 16. Let ABC be an equilateral triangle with side 2 inscribed in circle ω, and let P be a point on arc AB of its circumcircle. The tangent line to ω at P intersects lines AC and AB at E and F, respectively. If $PE = PF$, find EF.

Problem 17 (OMO Fall 2014/6). For an olympiad geometry problem, Tina wants to draw an acute triangle whose angles each measure a multiple of 10°. She doesn't want her triangle to have any special properties, so none of the angles can measure 30° or 60°, and the triangle should definitely not be isosceles.

How many different triangles can Tina draw? (Similar triangles are considered the same.)

Problem 18 (Alcumus). Six copies of the parabola $y = x^2$ are arranged (e.g. rotated) in the plane so that each vertex is tangent to the circle, and each parabola is tangent to its two neighbors. Find the radius of the circle.

Problem 19 (Mikhail Mishustin). Let AB be the diameter of a circle and C be another point on the circle. Construct the perpendicular from C to AB with only an unmarked straightedge.



Problem 20 (Spanish MO 1997/3). For each parabola $y = x^2 + px + q$ which intersects the coordinate axes in three different points, consider the circle passing through these three points. Prove that all these circles pass through a common point.

Problem 21 (Mandelbrot 2011). Let A and B be points on the lines $y = 3$ and $y = 12$, respectively. There are two circles passing through A and B that are also tangent to the x-axis, say at points P and Q. Suppose that $PQ = 2012$. Determine distance AB.

4 Number Theory

Problem 22. I'm thinking of a positive integer less than 5000. Here are some clues:

- When divided by 10, it leaves a remainder of 9.
- When divided by 9, it leaves a remainder of 8.
- . . .
- When divided by 2, it leaves a remainder of 1.

What is my number?

Problem 23 (Folklore). Can the average of two consecutive prime numbers be a prime number?