Diagram Perturbation

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In this handout we'll be manipulating geometric figures to discover some underlying properties. The manipulations are commonly known as transformations, but we'll be dealing with harder applications of such manipulations. Knowledge of angle chase and trigonometry is useful (and in some cases, necessary). Thanks to Evan Chen and CJ Quines for some of these problems.

Thanks to Sahil Chowdhury for proofreading.

Contents

1 Introduction

1.1 What is diagram perturbation?

The idea of ***diagram perturbation*** is to manipulate a diagram in a geometry problem in some way so that the result becomes easier to find. Let's try a few classic examples.

1.2 Rotations

Example 1
If a point P lies in an equilateral triangle ABC such that $AP = 3, BP = 4, CP = 5$, find the area of $\triangle ABC$.

Walkthrough. Rotate $\triangle APB$ around B such that A goes to C.

1. Find $\angle PBP'$.
2. Prove that $\triangle BPP'$ is equilateral and $\triangle PP'C$ is a right triangle
3. Find $\angle APB$.
4. Use LoC on $\triangle APB$ to find AB.
5. Finish.

We used a few strategies here:

- rotations (***cutting and pasting***),
- angle chasing, and
- trigonometry.

We will focus on rotations and similar ideas in this handout – in other words, rotations are an example of diagram perturbation.

> *Remark 2.* You will see a common pattern throughout problems of this type:
>
> - do something smart with the figure (i.e. diagram perturbation),
> - find some angles, and
> - apply length bashing techniques (e.g. trigonometry) to finish the problem.

Let's try one more example.

1.3 Reflections

I don't remember exactly how this was stated, but it's a problem any math enthusiast has heard of.

Example 3 (Folklore)
A man is at point A and wants to go to point B. However, he must first go to a river to get water, which is effectively a straight line. Note that points A and B are on the same side of the river. If he must go from A to a point on the river then to B, what is the path he should take?

Walkthrough. Reflect B across the river line to get B'.

1. Prove that this problem is equivalent to minimizing $AP + BP$, where P is the point on the river he goes to.
2. Show that $AP + BP = AP + B'P$.
3. Demonstrate $AP + B'P$ is minimized when P is the intersection of the river and AB'.

> *Remark 4 (Contest Example).* Putnam 1998/B2 is another type of this problem.

With rotations and reflections explained, let's move on to some harder ideas.

1.4 Isn't this just transformations?

You might be asking yourself, "I've known about rotations, reflections, translations, and dilations since 6th grade. What's different here?"

This is a good question. There actually is **no difference** (i.e. I won't magically pull out a new type of transformation from thin air). However, I've chosen to call this *diagram perturbation* as opposed to *transformations* because we aren't just reflecting the whole object. Finding what pieces to perturb and what auxiliary lines to draw is wildly harder than moving all the pieces. We're going to focus on drawing extra lines or moving parts of the figure in this handout.

2 Parallelograms

Parallelograms are effectively just reflecting a triangle across one of its midpoints. Let's take advantage of the angles formed.

Example 5
Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$ in triangle ABC. Prove $MN = \frac{1}{2}BC$ without using similar triangles.

Walkthrough. Let L be the midpoint of BC.

1. Prove $MN \parallel LC$ and $NC \parallel ML$. Conclude $MNCL$ is a parallelogram.
2. Finish by showing $MN = LC = LB$.

Example 6
Let M be the midpoint of $\overline{BC}$ in a triangle ABC. Given that $AM = 2$, $AB = 3$, $AC = 4$, find the area of $\triangle ABC$.

Walkthrough. Reflect A across M to get A'.

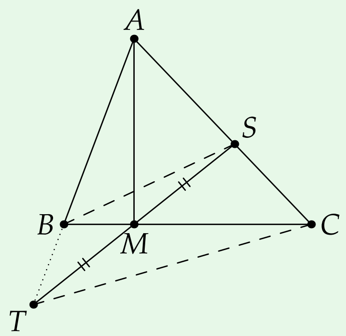
1. Prove that $ACA'B$ is a parallelogram.
2. Prove that $\triangle AA'B$ is isosceles and use it to find $[AA'B]$.
3. Use area relations to find $[ABC]$.

Example 7
A triangle ABC has medians of lengths m_a, m_b, m_c. Find the ratio of the area of the triangle formed by these medians to the area of triangle ABC.

Walkthrough. Let l be the line through A parallel to BC, and let D, E, F be the midpoints of BC, CA, AB respectively. Furthermore, let A' be a point on l such that $AA' = EF$.

1. Use parallelograms to show that $\triangle A'CF$ is a triangle formed by the medians of $\triangle ABC$.
2. Finish by area relations.

Example 8 (NIMO 8.8)
The diagonals of convex quadrilateral $BSCT$ meet at the midpoint M of $\overline{ST}$. Lines BT and SC meet at A, and $AB = 91$, $BC = 98$, $CA = 105$. Given that $\overline{AM} \perp \overline{BC}$, find the positive difference between the areas of $\triangle SMC$ and $\triangle BMT$.



Walkthrough.

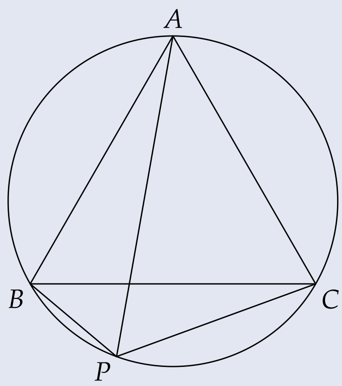
1. Find $\sin \angle BAM$ and $\sin \angle CAM$.
2. Let A_1 be the reflection of A over M. Use parallelograms to show $[AST] = [AA_1T]$.
3. Use trigonometry to find $[AA_1T]$ and finish.

3 Equilateral Triangles

3.1 Same Point on the Same Side

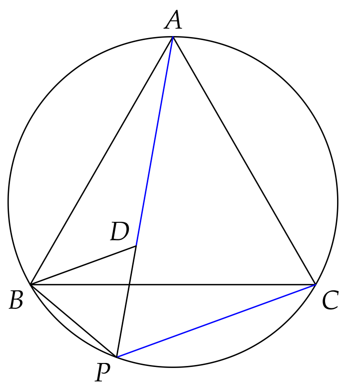
The idea is to split up a segment into two parts, or you can also think of it as adding two segments and seeing if that new segment can be found in the figure.

Theorem 9 (van Schooten's Theorem)
Let P be a point on the minor arc BC of equilateral triangle ABC. Then $PA = PB + PC$.



There is a quick solution using Ptolemy's theorem by applying it to quadrilateral $ABPC$. We'll try to prove this theorem without using Ptolemy's.

Proof. To prove that PA is the sum of PB and PC, let's try to split up PA into two segments. One will have length PB, and the other should have length PC. We pick the point D on segment PA such that $PD = PB$.



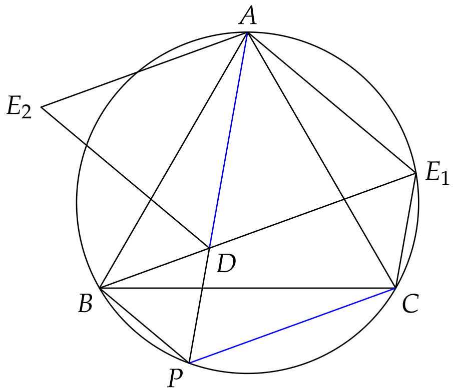
From construction, we have $PD = PB$, so to finish, we need to prove $AD = PC$. Let's look at what we can get from $PD = PB$. If we draw BD, it doesn't just look like triangle BDP is isosceles, but it looks like it's equilateral too.

Angle chasing, we get

$$\angle BPD = \angle BPA = \angle BCA = 60^\circ.$$

Since $\triangle BDP$ is isosceles, the remaining two angles are 60°, making it equilateral.

We can show that $AD = PC$ by constructing a similar equilateral triangle. Let E be the point such that $\triangle ADE$ is equilateral. But there's a problem: there are *two* possible choices of E on opposite sides of AD. Let's draw both and see what happens.



Surprisingly, it looks like E_1 lies on the circumcircle. It even looks like B, D, and E_1 are collinear! It also seems that this line is parallel to PC, which would make DE_1CP is a parallelogram. In fact, if it was a parallelogram, we'd be done.

Let's prove the following claims:

Claim — If DE_1CP is a parallelogram, then $AD = PC$.

Proof. Note that $PC = DE_1$. But $DE_1 = AD$ because $\triangle ADE_1$ is equilateral, and this finishes the proof. □

Claim — DE_1CP is a parallelogram.

Proof. Four steps:

- **B, D and E_1 are collinear**: $\angle ADE_1 = 60^\circ = \angle BDP$.
- **E_1 lies on the circumcircle**: $\angle AE_1B = \angle AE_1D = 60^\circ = \angle ACB$.
- **DE_1 and PC are parallel**: $\angle ADE_1 = 60^\circ = \angle ABC = \angle APC$.
- **DP and E_1C are parallel**: $\angle BE_1C = \angle BAC = \angle PAE_1 = 180^\circ - \angle PCE_1$.

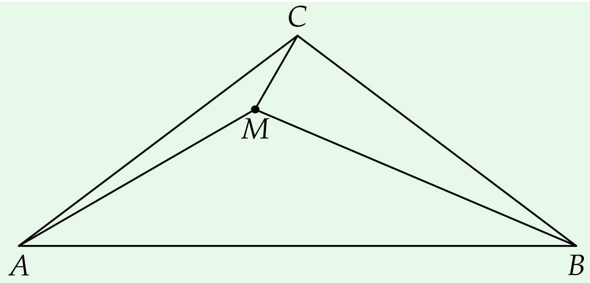
□

This gives us the desired result. □

3.2 Reflections

Let's just dive into an example:

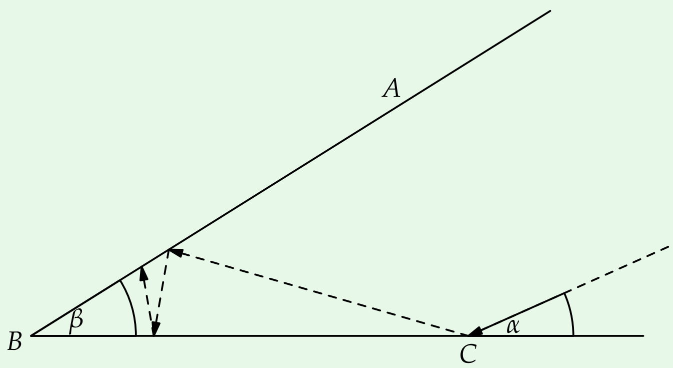
Example 10 (AIME I 2003/10)
Triangle ABC is isosceles with $AC = BC$ and $\angle ACB = 106^\circ$. Point M is in the interior of the triangle so that $\angle MAC = 7^\circ$ and $\angle MCA = 23^\circ$. Find the number of degrees in $\angle CMB$.



Walkthrough. Reflect M across the perpendicular from C to AB to get N.

1. Find $\angle CBN$ and $\angle BCN$.
2. Find $\angle MCN$ using the above step.
3. Prove $\triangle AMC \cong \triangle BNC$, i.e. $CM = CN$.
4. Use the above two steps to prove $\triangle CMN$ is equilateral.
5. Show by angle chase that $\triangle MNB \cong \triangle CNB$.
6. Finish.

Example 11 (AIME 1994/14)
A beam of light strikes $\overline{BC}$ at point C with angle of incidence $\alpha = 19.94^\circ$ and reflects with an equal angle of reflection as shown. The light beam continues its path, reflecting off line segments $\overline{AB}$ and $\overline{BC}$ according to the rule: angle of incidence equals angle of reflection. Given that $\beta = \alpha/10 = 1.994^\circ$ and $AB = BC$, determine the number of times the light beam will bounce off the two line segments. Include the first reflection at C in your count.



Walkthrough. At each point of reflection, we pretend instead that the light continues to travel straight.

1. After k reflections (excluding the first one at C), what angle does the extended line form at point B?
2. For the kth reflection to be just inside or at point B, bound $k\beta$ with α.
3. Solve for the largest possible value at k.
4. Remember to add 1 for the first reflection at C.

4 Translations

We've seen how rotations and reflections are useful. What about translations? In this case, we're just going to move **one point** and see how everything else follows. Here's a problem I came up with to demonstrate this:

Example 12
An equilateral triangle ACK is located inside a regular decagon $ABCDEFGHIJ$. If the area of the decagon is 2020, find the area of $HIJAK$.

Walkthrough. Let O be the center of the regular decagon.

1. Show that $[HAK] = [HAO]$. (Hint: prove $AH \parallel BG$.)
2. Use the above step to show $[HIJAK] = [HIJAO]$, and

$$[HIJAO] = \frac{3}{10}[ABCDEFGHIJ].$$

Remark 13. The idea is to take advantage of **same base-same height** (if the heights and bases are equal in length for two triangles, their areas are the same). This is the basis of moving a point, i.e. translation.

5 An IMO Shortlist Teaser

This is going to use the trick of creating a segment out of the sum of two segments.

We are **not** going to solve this problem! This is just a hint as to how to solve it. A solution is given here if you'd like to see the rest – we are just going to examine the construction. (The other part, Ceva, is a part left to the reader to prove.)

Example 14 (ISL 2000/G3)
ABC is an acute-angled triangle with orthocenter H and circumcenter O. Show that there are points D, E, F on BC, CA, AB respectively such that $OD + DH = OE + EH = OF + FH$ and AD, BE, CF are concurrent.

The constraint $OD + DH = OE + EH = OF + FH$ is extremely weird. It is well known that H and O share some nice connections. How could this help us?

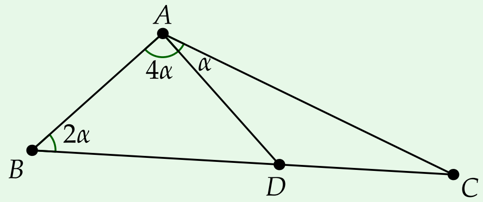
In a triangle ABC, if we reflect the orthocenter H across BC, we get a point H_A that lies on the circumcircle. Thus, $HD = H_AD$. But wait a minute! What if we connect O to H_A? What if we let D be the intersection of OH_A and BC?

We realize that $R = OD + DH_A = OD + DH$! This tells us that our weird constraint is actually just saying that their sum is equal to R. From here we can apply the other constraint and use Ceva to solve the problem.

6 Other Examples

Shamelessly taken from tkalid.

Example 15
As shown in the diagram below, $\angle DBA = 2\alpha$, $\angle BAD = 4\alpha$, and $\angle DAC = \alpha$. Given $AB = CD$, find α.

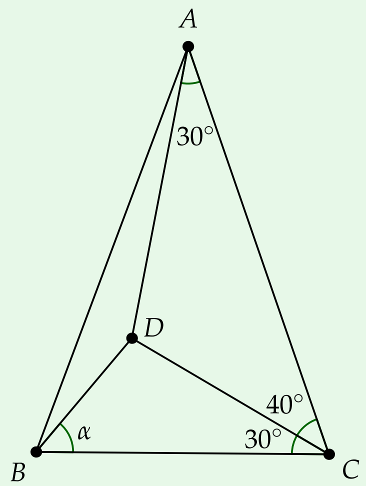


Walkthrough. Let E be the point on AC such that BE is the angle bisector of $\angle ABC$.

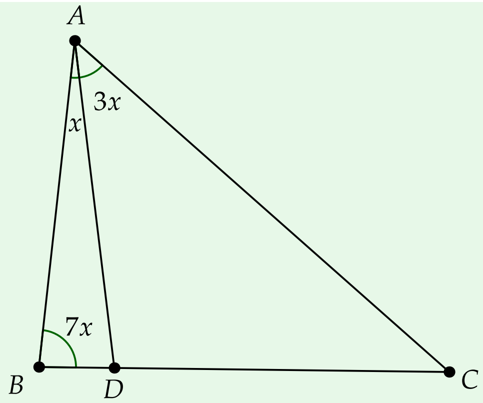
1. Find $\angle ABE$ and $\angle EBC$, and use them to show $\angle DAE = \angle DBE$.
2. Deduce ABD is cyclic. Use this to find $\angle ADE$.
3. Find $\angle EDC$. Use this to deduce $AE = ED$.
4. Show $\triangle EDC \sim \triangle EAB$. Use this to find $\angle DCE$.
5. Sum up the angles in $\triangle ABC$ and finish.

Do these next two on your own.

Example 16
As shown in the figure below, $\angle DAC = 30^\circ$, $\angle DCA = 40^\circ$, and $\angle DCB = 30^\circ$. Given that $AB = AC$, find α.



Example 17
In the diagram below, $\angle ABC = 7x$, $\angle BAD = x$, and $\angle DAC = 3x$. Given that $AD = BC$, find x.



7 Strategies

- **Symmetry**: take advantage of this. In particular, you can create symmetry by applying transformations.
- **Angle chasing**: use cyclic quadrilaterals and similarity to get some angles. Transformations can also help.
- **Auxiliary lines**: draw lines, because they help you find out what exactly you're missing.
  - **Parallelograms**: construct them when dealing with midpoints or, more obviously, parallel lines.
  - **Equilateral triangles**: if there is one in the figure, refer back to the bullet point above about symmetry. If there isn't, try applying transformations to find a hidden one.
- **Same point on the same side**: make the sum of two segments into a segment. If $AX + AY$ appears on one side, construct a point Y' on ray AX such that $XY' = AY$. Then $AX + AY = AY'$, and AY' hopefully makes an isosceles triangle, parallelogram, isosceles trapezoid, or cyclic quadrilateral. Note that you can try constructing on ray AY instead. Try to construct in the opposite direction.
- **Same point on opposite sides**: make the difference of two segments into a segment.
- **Transformations**: obviously, these are useful. But how?
  - **Rotations**: cut and paste a bit of the figure and attach it elsewhere. Usually, you want to attach it so that two sides line up because they have the same length.
  - **Reflections**: reflect isosceles figures.
  - **Translations**: try moving one point and see what happens. We take advantage of same base-same height here.
- **Length bashing**: this is mostly just used for answer extraction. However, sometimes bashing out that two lengths are the same is a good indication something interesting is occurring.

8 Problems

8.1 Classics

These are examples of Langley's problems that might serve better as brainteasers. Here is a generalized way to solve them.

1. In isosceles triangle ABC, $AB = AC$ and $\angle BAC = 20^\circ$. Points D and E are on AC and AB respectively such that $\angle CBD = 40^\circ$ and $\angle BCE = 50^\circ$. Determine $\angle CED$.

2. In isosceles triangle ABC, $AB = AC$ and $\angle BAC = 20^\circ$. Points D and E are on AC and AB respectively such that $\angle CBD = 50^\circ$ and $\angle BCE = 60^\circ$. Determine $\angle CED$.

3. In isosceles triangle ABC, $AB = AC$ and $\angle BAC = 20^\circ$. Points D and E are on AC and AB respectively such that $\angle CBD = 60^\circ$ and $\angle BCE = 70^\circ$. Determine $\angle CED$.

4. In convex quadrilateral $ABCD$, $\angle ABD = 12^\circ$, $\angle ACD = 24^\circ$, $\angle DBC = 36^\circ$, and $\angle BCA = 48^\circ$. Determine $\angle ADC$.

5. In convex quadrilateral $ABCD$, $\angle ABD = 38^\circ$, $\angle ACD = 48^\circ$, $\angle DBC = 46^\circ$, and $\angle BCA = 22^\circ$. Determine $\angle ADC$.

Have these shown up in contest? Yep!

Problem 1 (AMC 10B 2008/24). In convex quadrilateral $ABCD$, $AB = BC = CD$, $\angle ABC = 70^\circ$, and $\angle BCD = 170^\circ$. Determine $\angle DAB$.

8.2 Rotations

Problem 2 (AIME I 2012/13). Three concentric circles have radii 3, 4, and 5. An equilateral triangle with one vertex on each circle has side length s. The largest possible area of the triangle can be written as $a + \frac{b}{c}\sqrt{d}$, where a, b, c, and d are positive integers, b and c are relatively prime, and d is not divisible by the square of any prime. Find $a + b + c + d$.

Problem 3 (HMMT Feburary Geometry 2020/5). Let $ABCDEF$ be a regular hexagon with side length 2. A circle with radius 3 and center at A is drawn. Find the area inside quadrilateral $BCDE$ but outside the circle.

8.3 Reflections

Problem 4 (AMC 12A 2014/20). In $\triangle BAC$, $\angle BAC = 40^\circ$, $AB = 10$, and $AC = 6$. Points D and E lie on $\overline{AB}$ and $\overline{AC}$ respectively. What is the minimum possible value of $BE + DE + CD$?

Problem 5 (AMC 12A 2021/11). A laser is placed at the point $(3, 5)$. The laser beam travels in a straight line. Larry wants the beam to hit and bounce off the y-axis, then hit and bounce off the x-axis, then hit the point $(7, 5)$. What is the total distance the beam will travel along this path?

Problem 6 (BOGTRO Mock AMC 10 2014/18). Kelvin the frog enjoys hopping around on his infinite Cartesian plane. His house currently rests at the origin, and his friend Alex the Kat lives at $(5, 6)$. Kelvin comes over to Alex the Kat'z house every day to do USACO problems, but he must first always stop somewhere along the power line $y = 9$ to turn on the internet. What is the minimum possible distance that Kelvin can travel, first turning on the power and then ending up at Alex the Kat'z house?

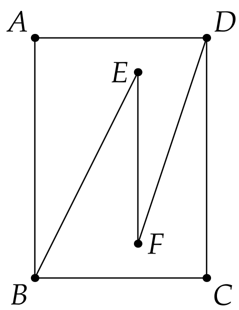
Problem 7 (NIMO 14.3). In triangle ABC, we have $AB = AC = 20$ and $BC = 14$. Consider points M on $\overline{AB}$ and N on $\overline{AC}$. If the minimum value of the sum $BN + MN + MC$ is x, compute $100x$.

Problem 8 (NIMO 1.7). Point P lies in the interior of rectangle $ABCD$ such that $AP + CP = 27$, $BP - DP = 17$, and $\angle DAP \cong \angle DCP$. Compute the area of rectangle $ABCD$.

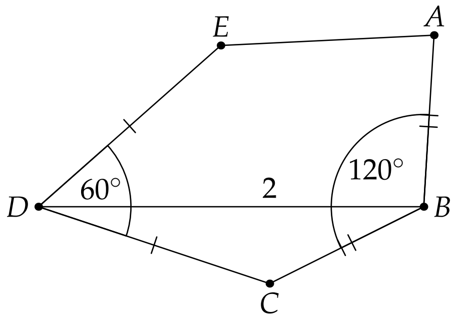
Problem 9 (HMMT November General 2019/9). Let $ABCD$ be an isosceles trapezoid with $AD = BC = 255$ and $AB = 128$. Let M be the midpoint of CD and let N be the foot of the perpendicular from A to CD. If $\angle MBC = 90^\circ$, compute $\tan \angle NBM$.

8.4 Parallelograms

Problem 10 (AIME 2011/2). In rectangle $ABCD$, $AB = 12$ and $BC = 10$. Points E and F lie inside rectangle $ABCD$ so that $BE = 9$, $DF = 8$, $\overline{BE} \parallel \overline{DF}$, $\overline{EF} \parallel \overline{AB}$, and line BE intersects segment $\overline{AD}$. The length EF can be expressed in the form $m\sqrt{n} - p$, where m, n, and p are positive integers and n is not divisible by the square of any prime. Find $m + n + p$.

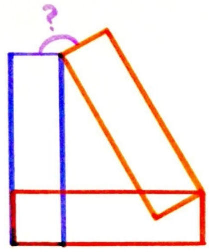


Problem 11. Let $ABCDE$ be a convex pentagon with $AB = BC$ and $CD = DE$. If $\angle ABC = 2\angle CDE = 120^\circ$ and $BD = 2$, find the area of $ABCDE$.



8.5 Equilateral Triangles

Problem 12 (Catriona Shearer). Find the angle labeled by the question mark.

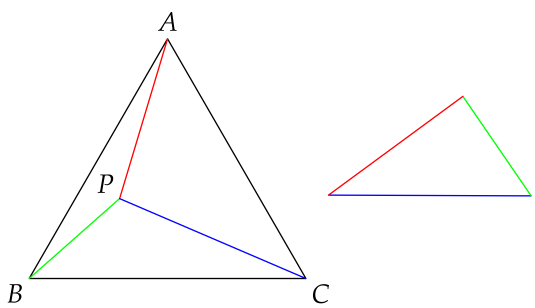


> *Remark 18.* For the above problem, rotations would also work.

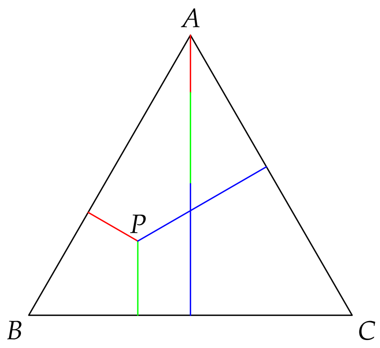
Problem 13 (AMC 10A 2021/21). Let $ABCDEF$ be an equiangular hexagon. The lines AB, CD, and EF determine a triangle with area $192\sqrt{3}$, and the lines BC, DE, and FA determine a triangle with area $324\sqrt{3}$. The perimeter of hexagon $ABCDEF$ can be expressed as $m = n\sqrt{p}$, where m, n, and p are positive integers and p is not divisible by the square of any prime. What is $m + n + p$?

Problem 14 (AMC 12A 2020/24). Suppose that $\triangle ABC$ is an equilateral triangle of side length s, with the property that there is a unique point P inside the triangle such that $AP = 1$, $BP = \sqrt{3}$, and $CP = 2$. What is s?

Problem 15 (Pompeiu's Theorem). Let P be a point *not* on the circumcircle of an equilateral triangle ABC. Then there exists a triangle with side lengths PA, PB, and PC.



Problem 16 (Viviani's Theorem). Let P be a point inside equilateral triangle ABC. Then the sum of the distances from P to the sides of the triangle is equal to the length of its altitude.

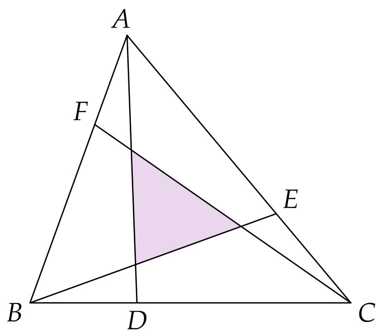


The following is not an equilateral triangle problem, but it is still has a similar idea nonetheless:

Problem 17 (One-Seventh Area Triangle). In triangle ABC, points D, E, and F lie on sides BC, CA, and AB respectively such that

$$\frac{CD}{BD} = \frac{AE}{CE} = \frac{BF}{AF} = 2.$$

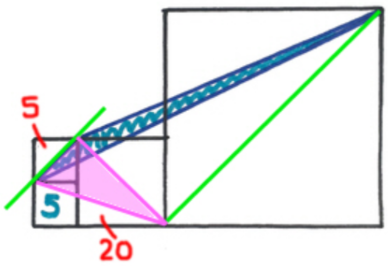
Then, the area of the inner triangle formed by the lines AD, BE, and CF is one-seventh the area of ABC.



> *Remark 19.* A generalization of the above is Routh's theorem.

8.6 Translations

Problem 18 (Catriona Shearer). In this figure there are four squares. The area of the two little squares is 5 and the area of the middle square is 20. What is the area of the blue triangle? (Note that the figure hints at the answer.)

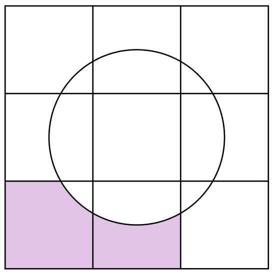


Problem 19 (MOP). Consider rectangle $ABCD$ with point M in its interior. If $\angle BMC + \angle AMD = 180^\circ$, find $\angle BCM + \angle DAM$.

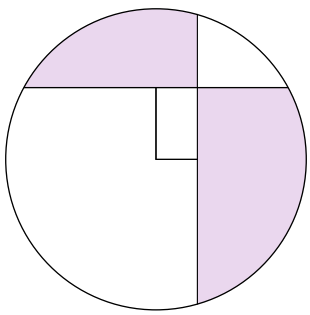
8.7 Miscellaneous

This is just a few problems that I thought were interesting. Have fun!

Problem 20. A circle with radius 1 is drawn centered on a 3×3 grid of unit squares. Find the area inside the lower-left and bottom squares but outside the circle.



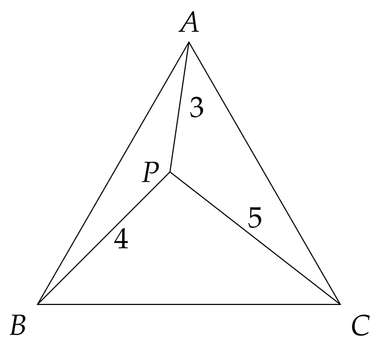
Problem 21 (UKMT 2014). A circle with area 2500 is divided by two perpendicular chords into four regions. The two regions next to the region with the circle's center, shaded in the figure, have combined area 1000. The center of the circle and the intersection of the chords form opposite corners of a rectangle, whose sides are parallel to the chords. What is the area of this rectangle?



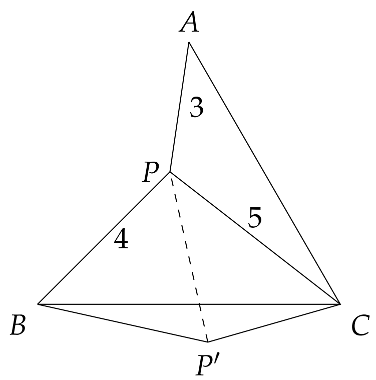
9 Selected Solutions

9.1 Solution 1

Let's draw a figure:



Note that $3, 4, 5$ are special numbers, because they form the side lengths of a right triangle. Now we're going to rotate $\triangle APB$ around B such that A goes to C:



Let's angle chase. Note that $\angle ABP = \angle CBP'$ and $\angle APB + \angle CBP = 60^\circ$, so $\angle PBP' = 60^\circ$. Combined with the fact that $BP = BP' = 4$, we must have that $\triangle BPP'$ is an equilateral triangle. Furthermore, $CP' = 3$, implying $\triangle PP'C$ is a $3-4-5$ right triangle. Thus,

$$\angle APB = \angle CP'B = \angle BP'P + \angle CP'P = 60^\circ + 90^\circ = 150^\circ,$$

and we can use Law of Cosines on $\triangle APB$ to get

$$AB^2 = 3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cdot \cos 150^\circ = 25 + 12\sqrt{3},$$

implying

$$[ABC] = \frac{AB^2\sqrt{3}}{4} = \boxed{\frac{25\sqrt{3}+36}{4}}.$$

9.2 Solution 3 (Folklore)

Let's reflect B across the river line to get B'. Thus, if the point on the river he goes to is P, then we are trying to minimize $AP + BP$. But $BP = B'P$ (since it is a reflection), so

$$AP + BP = AP + B'P.$$

Furthermore, the shortest distance from A to B' in general is just the line segment AB', and in this case, P would be the intersection of AB' and the river line. Thus, we reflect $B'P$ back across the line, and this is the path we should take.

9.3 Solution 5

Let L be the midpoint of BC. Then $MN \parallel LC$ and $NC \parallel ML$, implying $MNCL$ is a parallelogram. Thus, $MN = LC = LB$, and we're done.

9.4 Solution 6

Reflect A across M to get A'. Then AA' and BC bisect each other, implying $ACA'B$ is a parallelogram. Furthermore, $AM = MA' = 2$, so $BA' = A'A = 4$, which means $\triangle AA'B$ is isosceles. Thus,

$$[AA'B] = \frac{3\sqrt{55}}{4}.$$

We know that $[AA'B] = [AMB] + [MBA']$ and

$$[AMB] = [CMA] = [A'MC] = [BMA'],$$

so

$$[ABC] = [AA'B] = \boxed{\frac{3\sqrt{55}}{4}}.$$

9.5 Solution 7

Let l be the line through A parallel to BC, and let D, E, F be the midpoints of BC, CA, AB respectively. Furthermore, let A' be a point on l such that $AA' = EF$. We can easily prove through parallelograms that $\triangle A'CF$ is a triangle formed by the medians of $\triangle ABC$ (prove this yourself!). If we let S be one of the four equal areas formed by parallelogram $AA'EF$ and its diagonals, we have that $[AA'EF] = 4S$, and $[A'CF] = 6S$. Furthermore, since $[AEF] = 2S$, we have that $[ABC] = 8S$. Thus, the answer is

$$\frac{[AC'F]}{[ABC]} = \frac{6S}{8S} = \boxed{\frac{3}{4}}.$$

9.6 Solution 8 (NIMO 8.8)

Solution by Evan Chen.

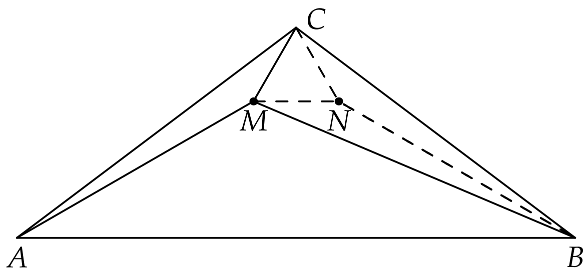
Let's get rid of B and C first. Set $\beta = \angle BAM$, $\gamma = \angle CAM$ and note that $\sin\beta = \frac{5}{13}$ and $\sin\gamma = \frac{3}{5}$. Compute $AM = 84$. Now, let A_1 be the reflection of A over M. We can compute

$$\begin{aligned}
[AST] &= [AA_1T] \\
&= \frac{AA_1^2 \sin\beta \sin\gamma}{2\sin(\beta+\gamma)} \\
&= 7^2 \cdot 288 \cdot \frac{\frac{5}{13}\cdot\frac{3}{5}}{\frac{5}{13}\cdot\frac{4}{5}+\frac{12}{13}\cdot\frac{3}{5}} \\
&= 7^2 \cdot 288 \cdot \frac{15}{56} \\
&= 3780.
\end{aligned}$$

In that case, the desired quantity is $[ABC] - [AST] = 84 \cdot 7^2 - 3780 = \boxed{336}$.

9.7 Solution 10 (AIME I 2003/10)

Let's reflect M across the perpendicular from C to AB to get N:



Then obviously $\angle CBN = 7^\circ$ and $\angle BCN = 23^\circ$. Thus,

$$\angle MCN = 106^\circ - 2 \cdot 23^\circ = 60^\circ.$$

Furthermore, since $\triangle AMC$ and $\triangle BNC$ are congruent by ASA, we must have that

$$CM = CN.$$

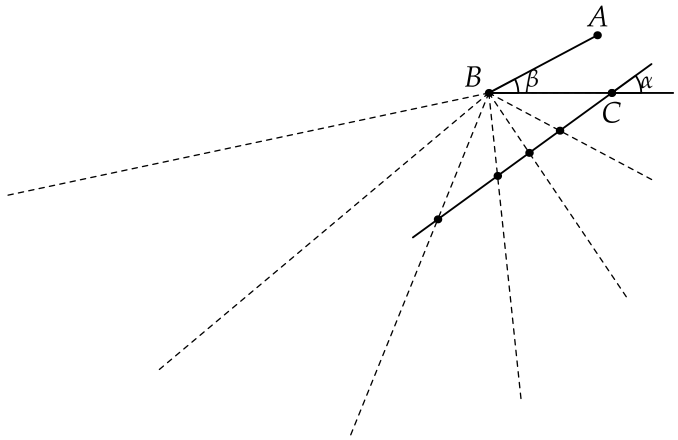
Hence $\triangle CMN$ is an equilateral triangle, so $\angle CNM = 60^\circ$. Thus

$$\angle MNB = 360^\circ - \angle CNM - \angle CNB = 360^\circ - 60^\circ - 150^\circ = 150^\circ.$$

We now see that $\triangle MNB$ and $\triangle CNB$ are congruent. Therefore, $CB = MB$, so $\angle CMB = \angle MCB = \boxed{83^\circ}$.

9.8 Solution 11 (AIME 1994/14)

At each point of reflection, we instead pretend that the light continues to travel straight.



Note that after k reflections (excluding the first one at C), the extended line will form an angle $k\beta$ at point B. For the kth reflection to be just inside or at point B, we must have $k\beta \le 180 - 2\alpha \implies k \le \frac{180-2\alpha}{\beta} = 70.27$. Thus, our answer is, including the first intersection, $\left\lfloor \frac{180-2\alpha}{\beta} \right\rfloor + 1 = \boxed{071}$.

9.9 Solution 12

Let O be the center of this decagon. Note that O lies on the line BG, as does K. Note that $AH \parallel BG$, so O to AH is the same distance as K to AH. Thus, by same base-same height, we must have $[AHO] = [AHK]$. Thus,

$$[HIJAK] = [HIJA] + [AHK] = [HIJA] + [AHO] = [HIJAO],$$

which is equivalent to three-tenths of the decagon (because $[HIJAO] = [HIO] + [IJO] + [JAO]$, each of which are congruent isosceles triangles, and the total area of the decagon is 10 of these equal isosceles triangles). Thus, the answer is

$$\frac{3}{10} \cdot 2020 = \boxed{606}.$$

9.10 Solution 15

Let E be the point on AC such that BE is the angle bisector of $\angle ABC$. Then we have $\angle ABE = \angle EBC = \alpha$. Connecting DE we see that $\angle DAE = \angle DBE$, so the quadrilateral $ABDE$ is cyclic, which means that $\angle ADE = \angle ABE = \alpha$. From this we get $\angle EDC = 5\alpha$. We know that $AE = ED$, so $\triangle EDC \sim \triangle EAB$, implying $\angle ABE = \angle DCE = \alpha$. Summing up the angles in $\triangle ABC$ we get $2\alpha + 5\alpha + \alpha = 180^\circ$, so $\alpha = \boxed{22.5^\circ}$.

9.11 Solution 16

Because $\triangle ABC$ is isosceles we know $\angle ABC = 70^\circ$ and $\angle BAC = 40^\circ$. Let the altitude from A to BC intersect DC at a point E. Then we know that $\triangle BEC$ is isosceles as well, so $\angle BED = 60^\circ$. We can also find $\angle DEA$. We know that $\angle EAC = 20^\circ$, so $\angle DEA = 40^\circ + 20^\circ = 60^\circ$. Now notice that in $\triangle BEA$, ED and AD are angle bisectors, so D is the incenter of $\triangle BEA$, and thus we know $\angle ABD = \angle DBE$. We also know that $\angle EBC = 30^\circ$, so $\angle EBA = 40^\circ$ and $\angle DBA = 20^\circ$. Finally, we have $\alpha = 70^\circ - 20^\circ = \boxed{50^\circ}$.

9.12 Solution 17

Let E be the point on AC such that $\angle ABE = 3x$. From this we get that $\triangle BEC \sim \triangle ABC$, so
$$\frac{BE}{AB} = \frac{BC}{AC}.$$
We know that $AD = BC$, so this is just $\frac{BE}{AB} = \frac{AD}{AC}$, or $\frac{AB}{AC} = \frac{BE}{AD}$. Therefore $\triangle BAE \sim \triangle ACD$ by SAS similarity, so $\angle DCA = \angle EAB = 4x$. Summing up the angles of $\triangle ABC$ we have $7x + 4x + 4x = 180$, so $x = \boxed{12}$.