Invariants

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In this handout we explore the basics of invariants and monovariants, including the application of techniques like AM-GM and weighting. Special thanks to Pranav Sriram's *Olympiad Combinatorics* and AMSP Combinatorics 3 for most of these problems.

Contents

1 Introduction

1.1 Definitions

Invariant
An ***invariant*** is a property or quantity that does not change under certain operations.

Monovariant
A ***semi-invariant*** or ***monovariant*** is a quantity that always increases or always decreases after the corresponding operation.

1.2 More Exposition

Classical examples of invariants are parity or algebraic expressions such as sums or products. Finding an invariant is a common idea in problems asking to prove that something cannot be achieved. Monovariants are also very efficient in showing that the corresponding process must stop after finitely many moves.

2 Classics

I'll skip over the "find the invariant and win" questions, since those just involve ***algebraic manipulation***. In other words, I'm skipping over to the main course.[1] In these problems we do one of three things (or a combination of them):

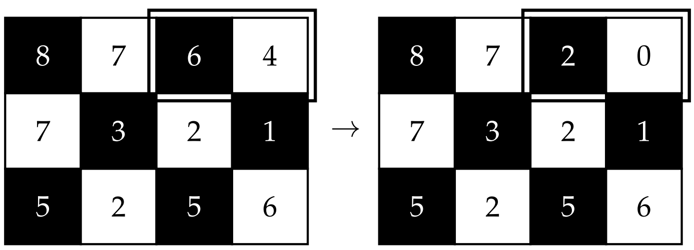
1. use algorithms, or
2. use modular arithmetic, or
3. use AM-GM.

Note AM-GM is for bounding.

Example 1 (ISL 1989)
A natural number is written in each square of an $m \times n$ chessboard. The allowed move is to add an integer k to each of two adjacent numbers in such a way that nonnegative numbers are obtained (two squares are adjacent if they share a common side). Find a necessary and sufficient condition for it to be possible for all the numbers to be zero after finitely many operations.

Walkthrough. Let S_b and S_w denote the sum of numbers on black and white squares, respectively.

1. Show that $S_b - S_w$ is invariant. The following figure might help:



[1]Try the problems in the problem set if you would like to see examples of these.

Thus, it seems that in order for all the numbers to be 0 at the end, we must have $S_b - S_w = 0$. We'll try to prove that this is the only thing we need to reach 0s at the end. Suppose a, b, c are numbers in cells A, B, C respectively, where A, B, C are cells such that A and C are both adjacent to B.

2. Find an algorithm for reducing a positive integer to 0 when $a \leq b$ and an algorithm for when $a \geq b$. Hint: play around with both cases using actual numbers.

3. Apply your algorithms in each row until all but the last two entries of each row are 0. Note that this just means that all columns besides the last 2 are filled with zeroes.

4. Apply the algorithm vertically until only two adjacent nonzero numbers remain. Clearly one of them is black square and the other is a white square.

5. Show that these two squares can only be cancelled if $S_b = S_w$ and finish.

This implies an important heuristic: **use an invariant to show a condition is necessary, and use an algorithm to show it's sufficient**.

Example 2 (ELMO 1999)
Jimmy moves around on the lattice point. From points (x, y) he may move to any of the points $(y, x), (3x, -2y), (-2x, 3y), (x+1, y+4)$ and $(x-1, y-4)$ show that if he starts at $(0, 1)$ he can never get to $(0, 0)$.

Walkthrough.

1. The sum of the coordinates of (x, y) in modulo 5 is $x + y \pmod 5$. What about after we apply the operations? (You'll have to consider each operation individually.)

2. Can the resulting sum of coordinates ever be $0 \pmod 5$ if we start with a sum of coordinates $1 \pmod 5$?

Always try invariants like:

1. sums

2. products

3. AM/GM/HM

This helps motivate what modulo is necessary.

Example 3 (ISL 2014)
The number 1 is written on each of 2^n sheets of paper. Each minute we are allowed to choose two distinct sheets, erase the two numbers a and b appearing on them and writing the number $a + b$ instead on both sheets. Prove that after $n2^{n-1}$ minutes the sum of the numbers on all sheets is at least 4^n.

Walkthrough. Let P be the product of the numbers on the sheets. Note that $P \geq 1$.

1. Show that by AM-GM,
$$(a+b)^2 \geq 4ab.$$
Alternatively, you can use trivial inequality.

2. After one operation, show the new product is $\frac{(a+b)^2}{ab}P \geq 4P$.

3. Thus, after $n2^{n-1}$ minutes the product is at least $4^{n \cdot 2^{n-1}}$.

4. Using AM-GM again, show that
$$\frac{S}{2^n} \geq \sqrt[2^n]{P}.$$

5. Manipulate the above to get $S \geq 4^n$ as desired.

Example 4
The numbers $1, 2, \ldots, 2008$ are written on a blackboard. Every second, Jimmy erases four numbers of the form $a, b, c, a+b+c$, and replaces them with the numbers $a+b, b+c, c+a$. Prove that this can continue for at most 10 minutes.

Walkthrough.

1. The obvious invariant: the sum never changes. The obvious monovariant: the number of terms decreases by 1 each second. Find the non-obvious invariant.

2. Let $x_1, x_2, \ldots, x_n$ be on the blackboard. Then by Cauchy-Schwarz,
$$n(x_1^2 + x_2^2 \ldots + x_n^2) \geq (x_1 + x_2 + \ldots + x_n)^2.$$
Plug in the two invariants to find the lower bound of n.

3. If there must be n terms on the blackboard, then at most $2008 - n$ seconds have passed. Show that our lower bound guarantees that at most 10 minutes have passed as desired.

Example 5 (St. Petersburg 2013)
There are 100 numbers from the interval $(0, 1)$ on the board. Every minute we can replace two numbers a, b on the board with the roots of $x^2 - ax + b = 0$ (if it has two real roots). Prove that this process must stop at some moment.

Walkthrough. Assume (for the sake of contradiction) the process is endless. Let $N < 1$ be a real number such that all of the 100 initial numbers are smaller than N.

1. Prove that if $a, b < N$ then
$$\frac{a + \sqrt{a^2 - 4b}}{2} < N.$$
Thus, all numbers on the board will always be smaller than N.

2. Let S and P be the sum and product of the numbers on the board, respectively. Furthermore, let S_0 and P_0 be the sum and product of the initial 100 numbers. Prove that
$$S < S_0 \qquad P > \frac{P_0}{N^M}.$$

3. Apply AM-GM to get $S \geq 100\sqrt[100]{P}$. Show that this gives us a contradiction for sufficiently large M.

Fact 6. When sums and products are used in invariant/monovariant questions, it is a good idea to use AM-GM.

Now for a monovariant:

Example 7 (ISL 2012)
Several positive integers are written in a row. Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y+1, x)$ or $(x-1, x)$. Prove that she can perform only finitely many such iterations.

Walkthrough. I'll let you do this one (basically) by yourself:

1. Show that
$$S = a_1 + 2a_2 + \ldots + ka_k + \ldots + na_n$$
is monovariant.
2. Hint: what is the difference between the new value of S and the old one after we perform a move?

3 Brutal Examples

Example 8 (ToT 2016)
On a blackboard several polynomials of degree 37 are written, each of them having leading coefficient equal to 1 and all coefficients nonnegative.

It is allowed to erase any pair of polynomials f, g and replace it by another pair of polynomials f_1, g_1 of degree 37 with leading coefficients to 1 such that either $f_1 + g_1 = f + g$ or $f_1 g_1 = fg$.

Can we reach a blackboard on which all polynomials have 37 distinct positive roots?

Walkthrough. Let
$$f = X^{37} + a_1 X^{36} + \ldots + a_{37}, \qquad g = X^{37} + b_1 X^{36} + \ldots + b_{37},$$
$$f_1 = X^{37} + c_1 X^{36} + \ldots + c_{37}, \qquad g_1 = X^{37} + d_1 X^{36} + \ldots + d_{37}.$$

1. Show that regardless of if we have $f_1 + g_1 = f + g$ or $f_1 g_1 = fg$, we will always have $a_1 + b_1 = c_1 + d_1$.
2. Show that this implies the sum of the coefficients of X^{36} of fg or $f+g$ is invariant.
3. Show that at every step, one of the two polynomials will have a nonnegative coefficient for X^{36}.
4. Use Vieta's to finish.

Fact 9. If we have more than one option as to what to turn the objects into (e.g. the ToT 2016 problem), it is often good to find a way to find an invariant that works for all options.

The following is not necessarily hard, but realizing the weighting is nontrivial.

Example 10
The first quadrant is divided into unit squares. We are allowed to perform the following move: if the square (x, y) has a token, while $(x, y+1), (x+1, y)$ are empty, then we take the token on (x, y) and put a token on each of the other two squares.

Initially, we have tokens on $(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)$. Can we clear these six squares by a sequence of moves?

Walkthrough. Assign the weight $\frac{1}{2^{x+y}}$ to the cell (x, y) if there is a token at (x, y) , and assign it a weight of 0 otherwise, for all positive integers x, y.

1. Prove that the sum of all the weights in the first quadrant is invariant. Furthermore, show that the sum is $\frac{11}{16}$.
2. If none of the six initial squares have a token, then what is the maximum sum of all weights? (Hint: maximum is achieved when we put weights in every eligible location.)
3. Show that this maximum sun can never equal $\frac{11}{16}$, contradiction.

Fact 11. Sometimes ***weighting*** is important for invariant questions, especially grid problems.

In general, when there's symmetry, attempt to weight the objects.

Example 12 (Russia 2014)
The polynomials $X^3 - 3X^2 + 5$ and $X^2 - 4X$ are written on the blackboard. If the polynomials $f(X)$ and $g(X)$ are written on the blackboard, we are allowed to write down the polynomials $f(X) \pm g(X), f(X) \cdot g(X), f(g(X))$ and $c \cdot f(X)$, where c is an arbitrary real constant. Can we write a nonzero polynomial of form $X^n - 1$ after a finite number of steps?

Walkthrough.

1. Show that if f' and g' have a common root z, then z is a common root of $(f \pm g)', (cf)', (f \cdot g)'$ and $(f \circ g)'$.
2. Find the common root of the derivatives of the initial two polynomials.
3. Show that the derivative of $X^n - 1$ cannot have a root of 2.

Example 13 (RMM SL 2016)
Start with any finite list of distinct positive integers. We may replace any pair $n, n+1$ (not necessarily adjacent in the list) by the single integer $n-2$, now allowing negatives and repeats in the list. We may also replace any pair $n, n+4$ by $n-1$. We may repeat these operations as many times as we wish. What is the most negative integer which can appear in a list?

Walkthrough. We'll try to artificially create an invariant.

1. Let ω be a solution to $x^n + x^{n+1} = x^{n-2}$. Show that it is also a solution to $x^n + x^{n+4} = x^{n-1}$. (Hint: divide by x^n.)

2. In any list, give the number n a weight of ω^n for any n appearing in that list. Show that the sum of the weights in the list at any step is invariant.

3. Show that
$$\sum_{n \in \mathbb{L}} x^n \leq x^{-4}.$$

4. Show that the above step implies there can't be any n's on the LHS that are less than or equal to -4.

5. Show that -3 can appear in a list.

Remark 14. The invariant in this problem is similar to the one in Conway's soldiers. The motivation behind this is ***recursion***, then transfer it to a ***characteristic polynomial***. Note that this is again a weighting problem.

Fact 15. For most invariant/monovariant questions, it is pretty easy to identify if the answer is yes or no (otherwise it wouldn't be a invariant/monovariant question!). The hard part is **proving** why your claim is true.

4 Problems

4.1 Appetizer

Problem 1. The cells of a 7×7 board are chess-painted (alternating colors) so that the corners are black. One is allowed to repaint any two adjacent cells to the opposite color. Is it possible to repaint the entire board white using such operations?

Problem 2. The numbers $1, 2, \ldots, 20$ are written on the board. One is allowed to erase any two numbers a and b and instead write the number $a + b - 1$. What number can remain on the board after 19 such operations?

Problem 3. Given a 1000-digit number with no zeroes, prove that from this number you can delete several (or none) last digits so that the resulting number is not a natural power less than 500 (a^1 is not considered a power).

Problem 4. The numbers 1 through 1000 are written on the board. One is allowed to erase any two numbers and and write the numbers ab and $a^2 + b^2$ instead. Is it possible with such operations to ensure that among the numbers written on the board, there are 700 at least that are the same?

Problem 5. Initially we have the numbers $\frac{49}{1}, \frac{49}{2}, \ldots, \frac{49}{97}$ on a board. A move consists in replacing two numbers, say a and b, with $2ab - a - b + 1$. After a series of moves, there is only one number left on the board. Find it!

4.2 Entree

Problem 6 (Russia 2008). A natural number is written on the blackboard. Whenever a number x is written, one can write either the number $2x + 1$ or $\frac{x}{x+2}$. At some point the number 2008 appears on the blackboard. Show that it was there from the beginning.

Problem 7 (Saint Petersburg 2020). The points $(1, 1), (2, 3), (4, 5)$ and $(999, 111)$ are marked in the coordinate system. If points (a, b) are marked then (b, a) and $(a - b, a + b)$ can be marked. If points (a, b) and (c, d) are marked then so can be $(ad + bc, 4ac - 4bd)$. Can we, after some finite number of these steps, mark a point belonging to the line $y = 2x$?

Problem 8 (Tuymaada Junior 2018). The numbers $1, 2, 3, \ldots, 1024$ are written on a blackboard. They are divided into pairs. Then each pair is wiped off the board and non-negative difference of its numbers is written on the board instead. 512 numbers obtained in this way are divided into pairs and so on. One number remains on the blackboard after ten such operations. Determine all its possible values.

4.3 Dessert

Full yet?

Problem 9. The numbers $1, 2, \ldots, n$ are written on a blackboard. Each minute, a student goes up to the board, chooses two numbers x and y, erases them, and writes the number $2x + 2y$ on the board. This continues until only one number remains. Prove that this number is at least $\frac{4}{9}n^3$.

Problem 10. Let n be a fixed positive integer. Initially, n 1's are written on a blackboard. Every minute, David picks two numbers x and y written on the blackboard, erases them, and writes the number $(x+y)^4$ on the blackboard. Show that after $n-1$ minutes, the number written on the blackboard is at least $2^{\frac{4n^2-4}{3}}$.

Problem 11 (USAMO 2019/5). Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime positive integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Evan can write 1 on the board in finitely many steps.

Problem 12 (Russia 2017). Initially a positive integer n is on the blackboard. Every minute we are allowed to take a number a on the blackboard, erase it and write instead all divisors of a except for a. After some time there are n^2 numbers on the blackboard. For which n is this possible?

Problem 13 (Iran RMM TST 2020). A 9×9 table is filled with zeroes. At every step we can either take a row, add 1 to every cell and shift it one unit to the right (the rightmost number in that row ends up in the leftmost position of the row) or take a column, subtract 1 from every number on that column and shift it one cell down (with the same convention as for rows). Can the table with the top right -1 and bottom left $+1$ and all other cells zero be reached?

5 Selected Solutions

5.1 Solution 1 (ISL 1989)

The following solution is by Pranav Sriram:

Note that in each move, we are adding the same number to 2 squares, one of which is white and one of which is black (if the chessboard is colored alternately black and white). If S_b and S_w denote the sum of numbers on black and white squares, respectively, then $S_b - S_w$ is an invariant. Thus if all numbers are 0 at the end, $S_b - S_w = 0$ at the end and hence $S_b - S_w = 0$ in the beginning as well. Thus, this condition is necessary; now we prove that it is sufficient.

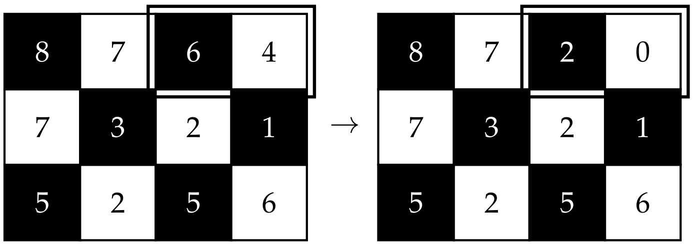


Figure 1: A move on the $m \times n$ board

Suppose a, b, c are numbers in cells A, B, C respectively, where A, B, C are cells such that A and C are both adjacent to B. If $a \leq b$, we can add $(-a)$ to both a and b, turning a to 0. If $a \geq b$, then add $a - b$ to b and c. Then b becomes a, and now we can add $-a$ to both of them, making them 0. Thus we have an algorithm for reducing a positive integer to 0. Apply this in each row, making all but the last 2 entries 0. Now all columns have only zeroes except the last two. Now apply the algorithm starting from the top of these columns, until only two adjacent nonzero numbers remain. These two numbers must be equal since $S_b = S_w$. Thus we can reduce them to 0 as well.

5.2 Solution 2 (ELMO 1999)

Let us take mod 5 of $x + y$. Note that since

$$\begin{aligned} 3x - 2y &\equiv 3(x+y) \pmod 5, \\ -2x + 3y &\equiv 3(x+y) \pmod 5, \\ x + 1 + y + 4 &\equiv x + y \pmod 5, \\ x - 1 + y - 4 &\equiv x + y \pmod 5, \end{aligned}$$

the sum of the two coordinates is either constant or multiplied by 3. Thus, $(0, 0)$ cannot be achieved.

5.3 Solution 3 (ISL 2014)

Consider the product P of the numbers on the sheets. Say we choose a, b and replace them by $a + b, a + b$. The quotient between the product of all numbers after the operation and the one before the operation is $\frac{(a+b)^2}{ab}$, which by AM-GM is greater than or equal to 4. Thus, the product is at least $4^{n \cdot 2^{n-1}}$, and by AM-GM again, we get that the sum S is

$$\left(\frac{S}{2^n}\right)^{2^n} \geq P,$$

implying the desired result $S \geq 4^n$.

5.4 Solution 4

Note that $a+b+c+(a+b+c) = (a+b)+(b+c)+(c+a)$. Thus, for every operation he does, the sum is constant, but the number of terms decreases by 1. Even more important,

$$a^2+b^2+c^2+(a+b+c)^2 = (a+b)^2+(b+c)^2+(c+a)^2,$$

implying the sum of squares is also invariant. Let $x_1, x_2, \ldots, x_n$ be on the blackboard. Then by Cauchy-Schwarz,

$$n(x_1^2+x_2^2 \ldots + x_n^2) \geq (x_1+x_2+\ldots+x_n)^2.$$

Taking into account our two invariants, we obtain

$$n \geq \frac{(1+2+\ldots+2008)^2}{1^2+2^2+\ldots+2008^2} = 1506 + \frac{502}{1339},$$

implying this can take place at most $2008-1506 = 502$ times, which is less than 600 seconds, or 10 minutes.

5.5 Solution 5 (St. Petersburg 2013)

Suppose this process is endless. There exists a real number $N < 1$ such that all of the 100 initial numbers are smaller than N. If $a, b < N$, we get

$$\frac{a+\sqrt{a^2-4b}}{2} < N.$$

Thus, all numbers on the board will always be smaller than N. Denote S and P as the sum and product, respectively, of all numbers on the board. Furthermore, let S_0 and P_0 be the sum and product of the initial 100 numbers. Each move gives us $S \to S-b$ and $P \to \frac{P}{a} > \frac{P}{N}$. After M moves, we get that $S < S_0$ and $P > \frac{P_0}{N^M}$. By AM-GM, we get $S \geq 100\sqrt[100]{P}$. This gives us a contradiction for sufficiently large M, implying the desired result.

5.6 Solution 7 (ISL 2012)

Clearly the maximum number on the board does not change, say it was M initially. Let $a_1, a_2, \ldots, a_n$ be the numbers on the board. I claim the quantity

$$S = a_1 + 2a_2 + \ldots + ka_k + \ldots + na_n$$

always increases. Say that at some moment $x > y$ with x to the left of y and x in position i. The difference between the new value of S and the old one is either

$$i(y+1)+(i+1)x-(ix+(i+1)y) = x-y+i \geq 1$$

or

$$i(x-1)+(i+1)x-(ix+(i+1)y) = -i+(i+1)(x-y) \geq -i+i+1 = 1,$$

proving the claim and the desired result.

5.7 Solution 8 (ToT 2016)

Suppose that

$$(X^{37} + a_1 X^{36} + \ldots + a_{37})(X^{37} + b_1 X^{36} + \ldots + b_{37})$$
$$= (X^{37} + c_1 X^{36} + \ldots + c_{37})(X^{37} + d_1 X^{36} + \ldots + d_{37}).$$

Looking at the coefficient of X^{36+37} we obtain

$$a_1 + b_1 = c_1 + d_1.$$

This also holds if

$$(X^{37} + a_1 X^{36} + \ldots + a_{37}) + (X^{37} + b_1 X^{36} + \ldots + b_{37})$$
$$= (X^{37} + c_1 X^{36} + \ldots + c_{37}) + (X^{37} + d_1 X^{36} + \ldots + d_{37}).$$

Thus, the sum of coefficients of X^{36} is invariant. Since initially all coefficients are nonnegative, at each step the sum of coefficients of X^{36} stays nonnegative. This implies that at least one polynomial, say $P(X) = X^{37} + a_1 X^{36} + \ldots + a_{37}$, has $a_1 \geq 0$ at every step. If $x_1, x_2, \ldots, x_{37}$ are the complex roots of P, then

$$x_1 + x_2 + \ldots + x_{37} = -a_1 \leq 0.$$

Thus, the polynomial cannot have 37 positive roots, which implies the desired result.

5.8 Solution 10

Consider the sum of $\frac{1}{2^{x+y}}$ over all pairs (x, y) for which there is a token at (x, y). Thus,

$$\frac{1}{2^{x+(y+1)}} + \frac{1}{2^{(x+1)+y}} = \frac{1}{2^{x+y}},$$

implying the sum is invariant. Initially, the sum is

$$\frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} = \frac{11}{16}.$$

Suppose that at some moment the six initial squares have no token. Then the sum is at most

$$\sum_{x,y \geq 1} \frac{1}{2^{x+y}} - \frac{11}{16} = 1 - \frac{11}{16} = \frac{5}{16},$$

which is a contradiction, proving the desired result.

5.9 Solution 12 (Russia 2014)

Let $f(X) = a_0 + a_1 X + \ldots + a_n X^n$, then its derivative is

$$f'(x) = a_1 + 2a_2 X + \ldots + n a_n X^{n-1}.$$

This satisfies all conditions, because

$$(f \pm g)' = f' \pm g',$$
$$(cf)' = cf',$$
$$(f \cdot g)' = f' \cdot g + g' \cdot f,$$
$$(f \circ g)' = (f' \circ g) \cdot g'.$$

Thus, if f' and g' have a common root z, then z is a common root of $(f \pm g)'$, $(cf)'$, $(f \cdot g)'$ and $(f \circ g)'$. The derivatives of the initial polynomials are $3X^2 - 6X$ and $2X - 4$, and 2 is a common root. However, $(X^n - 1)' = nX^{n-1}$ does not have the root $X = 2$, implying we can never get $X^n - 1$.

5.10 Solution 13 (RMM SL 2016)

Let's look for an invariant of the form $\sum_{n \in \mathbb{L}} x^n$, where $\mathbb{L}$ is a subset of $\mathbb{Z}$. To have an invariant, we want

$$x^n + x^{n+1} = x^{n-2},$$
$$x^n + x^{n+4} = x^{n-1},$$

for all n. This reduces to

$$x^2 + x^3 = 1,$$
$$x^5 + x = 1,$$

which is easily solvable since they are secretly the same equation, because

$$x^5 + x - 1 = (x^3 + x^2 - 1)(x^2 - x + 1).$$

Thus, we choose ω such that $\omega^3 + \omega^2 = 1$ and get $\sum_{n \in \mathbb{L}} \omega^n$ is constant, where $\mathbb{L}$ is the list at any step. Thus,

$$\sum_{n \in \mathbb{L}} \omega^n \leq \sum_{n \geq 1} \omega^n = \frac{\omega}{1 - \omega} = \omega^{-4}.$$

This must be true for all steps, and since $0 < \omega < 1$, we know that $n > -4$. Working backwards from -3, we find that an initial list of $1, 2, 3, 4, 5$ will generate -3, implying $\boxed{-3}$ works.