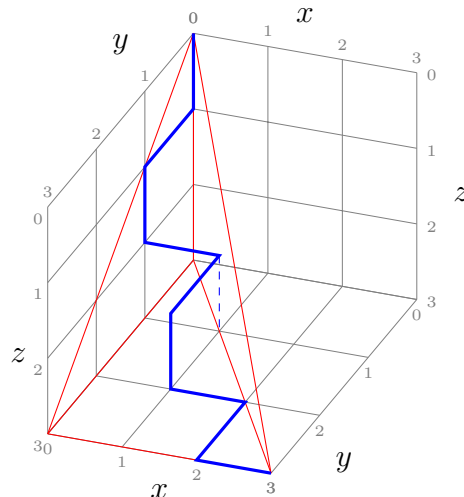# Recursion in the AMC and AIME

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*For:*
AoPS

*Date:*
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*The Catalan sequence in 3D. We will **not** be going over this.*

*To loop is human, to recurse is divine - Anonymous*

The purpose of this handout is to introduce recursion and how to use it in the AMC 10/12 and the AIME. This handout will also include practice problems at the end to help the reader learn how to use recursion. It should be noted that recursion is not the only strategy to use in a counting problem, but it is incredibly helpful to problems that can not be directly counted. If this document is missing anything or contains any mistakes, feel free to shoot me an email at jeffreychen51@gmail.com or dylanyu66@gmail.com.
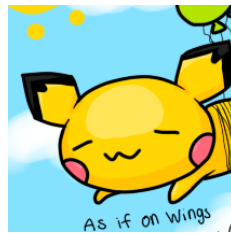
# Contents

# §0 Acknowledgements

This was made for the Art of Problem Solving Community out there! We would like to thank Evan Chen for his evan.sty code. In addition, all problems in the handout were either copied from the Art of Problem Solving Wiki or made by ourselves.



Art of Problem Solving Community



Evan Chen's Personal Sty File



FREEMAN66's Website - Say Hi!

And Evan says he would like this here for evan.sty:

```
Boost Software License - Version 1.0 - August 17th, 2003
Copyright (c) 2020 Evan Chen [evan at evanchen.cc]
https://web.evanchen.cc/ || github.com/vEnhance
```

Please do not make any copies of this document without referencing this original one.

# §1  Introduction to Recurrence

"To understand recursion, we must first understand recursion." The heart of recursion is to use the previous terms of the sequence to define future terms. It is a common technique to solve counting problems that traditional methods (e.g. the Principal of Inclusion-Exclusion, casework, and complementary counting) cannot solve. Instead of directly counting the number of objects, we use the first few terms, then try to find a relationship between these terms and the larger terms. This may sound confusing at first, but with enough practice, you will certainly be able to understand how and why it works. The best way to learn any sort of skill is with practice, and the same is true for recursion.

Suppose we have a sequence, $a_1, a_2, a_3, \cdots$, and assume we also have a relationship between terms of the sequence, such as

$$a_n = 3 \cdot a_{n-1}.$$

This is what is known as a recurrence relationship: expressing a term of a sequence as some particular function of the preceding terms. Taking our original recurrence relationship – $a_n = 3 \cdot a_{n-1}$, we can translate this to every term in the sequence is 3 times the last term in the sequence. If we let the first term, $a_1$, equal 1, then we can see that

$$a_2 = 3, a_3 = 9, a_4 = 27,$$

and so on. The important thing about the recurrence relationship is that the relationship is true for all positive integer $n$. For example, $a_{40} = 3 \cdot a_{39}$ just as $a_{100} = 3 \cdot a_{99}$. Because of this, we can make substitutions between the terms. For example, since $a_n = 3 \cdot a_{n-1}$ and $a_{n-1} = 3 \cdot a_{n-2}$, substituting the second equation into the first gives us $a_n = 9 \cdot a_{n-2}$.

Sticking with our recurrence relation of $a_n = 3 \cdot a_{n-1}$, and given that $a_1 = 1$, we can solve for the rest of the terms of the recurrence relation. We can make a table to calculate each value of $a_i$:

| Sequence of $a_n$ | | | | | | |
|---|---|---|---|---|---|---|
| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $a_i$ | 1 | 3 | 9 | 27 | 81 | 243 | 729 |

Looking at this, it seems like all $a_i$ are powers of 3. More specifically, it seems like $a_i = 3^{i-1}$.

> **Remark 1.1.** Trying a few cases can often give the desired explicit form.

We can prove this using **induction**, which will be introduced in our Proofs handout.

Given the first few terms of a recursion and it's recursive relationship, it is possible to get a formula for $a_i$. However, in short answer competitions such as the AMC and AIME, finding the formula usually isn't necessary. Just by using the recursive relationships, it is possible to compute $a_i$ by making a table and expressing $a_i$ as a function of the preceding terms. If you want to learn how to get an explicit formula for a general linear recurrence, this Wikipedia article explains it quite well. Here we will give a brief introduction.

# §2  Solving Linear Recurrences

> **Definition 2.1** (Characteristic Polynomial) — Let $x_1, x_2, \ldots$, be a sequence of real numbers. Consider a monic homogenous linear recurrence of the form
>
> $$x_n = c_{n-1}x_{n-1} + c_{n-2}x_{n-2} + \cdots + c_{n-k}x_{n-k}, \quad (*)$$
>
> where $c_1, \ldots, c_k$ are real constants. The **characteristic polynomial** of this recurrence is defined as the polynomial

$$P(x) = x^k - c_{k-1}x^{k-1} - c_{k-2}x^{k-2} - \cdots - c_1 x - c_k.$$

For example, let $F_n$ be the $n$th Fibonacci number defined by $F_1 = F_2 = 1$, and

$$F_n = F_{n-1} + F_{n-2} \iff F_n - F_{n-1} - F_{n-2} = 0,$$

then its characteristic polynomial is $x^2 - x - 1$.

The roots of the polynomial can be used to write a closed form for the recurrence.

---

**Theorem 2.2** (Explicit Form of a Linear Recurrence)

If the roots of the characteristic polynomial $P$ are distinct, then suppose the roots are $r_1, r_2, \cdots, r_k$. Then, there exists real constants $a_1, a_2, \cdots, a_k$ such that

$$x_n = a_1 \cdot r_1^n + a_2 \cdot r_2^n + \cdots + a_k \cdot r_k^n.$$

If there are roots with multiplicity greater than 1, suppose $r_1 = r_2 = \cdots = r_{k_1}$. Then we would replace the term $a_1 \cdot r^n + a_2 \cdot r_2^n + \cdots + a_{k_1} \cdot r_{k_1}^n$ with the expression

$$(a_1 \cdot n^{k_1 - 1} + a_2 \cdot n^{k_1 - 2} + \cdots + a_{k_1 - 1} \cdot n + a_{k_1}) \cdot r^n.$$

---

The proof of this theorem is by induction, which we are not discussing in this handout. **Binet's Formula** is proved using this idea of a characteristic polynomial:

---

**Theorem 2.3** (Binet's Formula)

Let $F_n = F_{n-1} + F_{n-2}$ and $F_0 = F - 1 = 1$. Then the explicit form is

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n\right).$$

---

*Proof.* We can replace all $F_k$ with $x^k$. Thus,

$$x^k = x^{k-1} + x^{k-2},$$

implying

$$x^2 - x - 1 = 0.$$

The solutions to this equation are

$$x_{1,2} = \frac{1 \pm \sqrt{1 + 4 \cdot (-1) \cdot (-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

Thus, the formula must be of the form

$$F_n = a_1 \left(\frac{-1 + \sqrt{5}}{2}\right)^n + a_2 \left(\frac{-1 - \sqrt{5}}{2}\right)^n,$$

for constants $a_1, a_2$. To find these two constants, we simply plug in $n = 0, 1$, and get the desired result. $\qquad \square$

Thus, we can see here that solving recursions is extremely powerful. There won't be many examples in this handout, but constructing problems is extremely easy.

## §3 First Counting Problem

While recursion might seem like a purely algebraic idea at first and something that looks fun but not useful, it can help solve many difficult counting problems (as well as certain computer science problems).

---

**Example 3.1**

Fred wants to climb a 10 step staircase. He can climb either 1 or 2 steps at a time. In how many ways can he climb the staircase?

---

*Solution.* We might try to count this directly. Perhaps we could list out all the possible combinations of how Fred could walk, but using traditional methods such as complementary counting, Principle of Inclusion-Exclusion, and casework will lengthen the time spent on this example. Instead, we will use recursion.

Instead of thinking of this problem as "Fred wants to climb 10 stairs", we will think of it more generally as "Fred wants to climb $n$ stairs." Then, we can define

$$a_n = \text{The number of ways for Fred to walk } n \text{ steps of stairs.}$$

With recursion in mind, we want to try to find a way to relate $a_n$ to the preceding terms. Think of the last step that Fred takes. This step can either be one stair or two stairs. If the last step is two stairs, then Fred would have climbed $n - 2$ stairs and then climbed two more stairs. The number of ways for Fred to climb $n - 2$ stairs is $a_{n-2}$ due to our definition of $a_i$, so the number of ways for Fred to climb $n - 2$ stairs and then climb 2 steps is $a_{n-2}$. If the last step is one stair, then Fred would have climbed $n - 1$ stairs and then climbed one stair. The number of ways for this to happen is $a_{n-1}$. Since these are the only two possibilities - either Fred's last step is one or two stairs, we have the following equation:

$$a_n = a_{n-1} + a_{n-2}$$

Now we can solve the problem. It is easy to see that $a_1 = 1$ as Fred can climb 1 stair in only 1 way, and $a_2 = 2$ because Fred can either climb 2 stairs in 2 one-stair steps or 1 two-stair step. We can make a table to solve for $a_{10}$:

| Sequence of $a_n$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $a_i$ | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 |

Thus, our answer is $\boxed{89}$.          □

Some of you may recognize this sequence as the Fibonacci sequence, which appears quite often in counting problems and especially ones that use recursion. Knowing the first few elements of the Fibonacci sequence can sometimes be handy for certain speed based competitions. As a side note, the Fibonacci numbers have a lot of interesting properties. For those unaware,

---

**Definition 3.2** (Fibonacci Sequence) — The **Fibonacci sequence** is a sequence of integers in which the first and second terms are both equal to 1 and each subsequent term is the sum of the two preceding it. The first few terms are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$.

---

The Fibonacci sequence can be written recursively as $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. This is the simplest nontrivial example of a linear recursion with constant coefficients.

## §4 Harder Recursion Problems

> **Example 4.1** (2007 AMC 12)
>
> Call a set of integers *spacy* if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, are spacy?

*Solution.* Counting directly seems to be somewhat difficult since there would be a lot of casework involved, so we think of recursion. We will define

$$a_n = \text{The number of spacy subsets of the set } \{1, 2, \ldots, n.\}$$

Next, we will try to relate $a_n$ to the terms preceding it. For each subset, we can put it into one of two groups: whether the subset contains $n$ or does not contain $n$. If the subset contains $n$, then because it is spacy, that subset cannot contain $n-1$ or $n-2$. Thus, this spacy subset is simply a spacy subset of $\{1, 2, \ldots, n-3\}$ with an appended $n$. Thus, there must be $a_{n-3}$ spacy subsets that contain $n$. If a spacy subset does not contain $n$, then it is also a spacy subset of $\{1, 2, \ldots, n-1\}$, so there are $a_{n-1}$ spacy subsets that do not contain $n$. Since a spacy subset either contains $n$ or does not contain $n$, we have exhausted all the possibilities. This results in the following recursive relationship:

$$a_n = a_{n-1} + a_{n-3}$$

We can calculate by hand the first few values of $a_i$. We have $a_1 = 2$ since the two spacy subsets are $\{\}$ and $\{1\}$. Similarly, $a_2 = 3$ since the spacy subsets are $\{\}, \{1\}, \{2\}$, and $a_3 = 4$ since the spacy subsets are $\{\}, \{1\}, \{2\}, \{3\}$. Now, with these values, we can make a table and solve for $a_{12}$:

| Sequence of $a_n$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| $a_i$ | 2 | 3 | 4 | 6 | 9 | 13 | 19 | 28 | 41 | 60 | 88 | 129 |

Thus, our final answer is $\boxed{a_{12} = 129}$.

> **Example 4.2** (AIME 2006)
>
> A collection of 8 cubes consists of one cube with edge-length $k$ for each integer $k$, $1 \le k \le 8$. A tower is to be built using all 8 cubes according to the following rules:
>
> - Any cube may be the bottom cube in the tower.
>
> - The cube immediately on top of a cube with edge-length $k$ must have edge-length at most $k+2$.
>
> How many towers can be constructed?

Again, we will solve this problem recursively. Define

$$a_n = \text{The number of different towers given } n \text{ cubes with edge-lengths } 1, 2, \ldots n$$

Then we look at the cube with edge-length $n$. This cube can either be on top of the cube with edge length $n-1$, on top of the cube with edge length $n-2$, or it could be at the bottom of the tower. Thus, for each configuration with $n-1$ cubes, there are 3 ways to create a configuration with $n$ cubes, and we have the following equation:

$$a_n = 3 \cdot a_{n-1}$$

Since $a_2 = 2$ (for all $n \ge 3$) as we can either have the cube with edge-length 1 on the bottom or the cube with edge-length 2 on the bottom, we have $a_8 = 3^6 \cdot 2 = \boxed{1458}$. $\qquad\square$

Observe that we can't say that $a_2 = 3 \cdot a_1$. This is because our recurrence relationship breaks down when $n = 2$. We assumed that the cube with edge-length $n$ can either be placed on the bottom, above the cube with edge-length $n - 1$, or above the cube with edge-length $n - 2$; however, when there is no cube with edge-length $n - 2$, (as when we substitute $n = 2$ this becomes 0), the recurrence is no longer $a_n = 3 \cdot a_{n-1}$. Sometimes for recurrences we can't directly recur from the very beginning and have to find the first few terms before the recurrence actually applies.
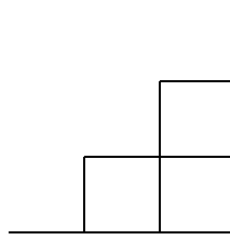
## §5 Catalan Numbers

The Catalan numbers are best presented through the following problem:

> **Example 5.1**
> How many $2n$ step paths are there from $(0,0)$ to $(n,n)$ such that for each $(x,y)$ on that path, $x \geq y$?

*Solution.* We can illustrate the problem for the $n = 3$ case here:



Counting by hand we can see that there are 5 paths. But we could try to do this recursively. We will define

$$C_n = \text{The number of paths from } (0,0) \text{ to } (n,n) \text{ such that for each } (x,y) \text{ on that path, } x \geq y.$$

Let the point $(i,i)$ be the first point in a path where $x = y$, besides the point $(0,0)$. Then, we can see that the number of ways to get from $(i,i)$ to $(n,n)$ is $C_{n-i}$. However, the number of ways to get from $(0,0$ to $(i,i)$ is $C_{i-1}$. Note that if $(i,i)$ is the first point in the path where $x = y$, then the path had to go from $(0,0)$ to $(1,0)$ to $(i,i-1)$ to $(i,i)$. Then, the portion of the path from $(1,0)$ to $(i,i-1)$ must always be below the line $x \geq y + 1$, as illustrated below:
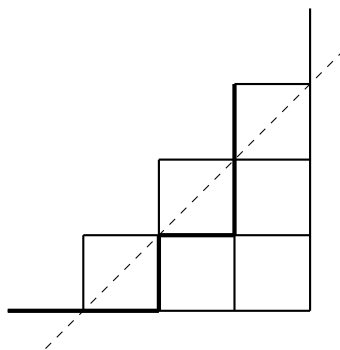


As we can see, the path never goes above the line $x = y + 1$ except at the very end. Then, the number of paths that goes from $(1,0)$ to $(i,i-1)$ that doesn't go above the line $x = y + 1$ has to be $C_i$ as we can just translate that portion of the path 1 unit to the left, and we get a graph from $(0,0)$ to $(i-1,i-1)$ that never goes above the line $x = y$. Therefore, the number of paths that intersect the line $y = x$ first at point $(i,i)$ will be $C_{i-1}C_{n-i}$. Summing this over all $i$, as the path can only touch the line $x = y$ first at exactly one point, we get the following recurrence:

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \ldots + C_{n-2} C_1 + C_{n-1} C_0 = \sum_{i=0}^{n-1} C_i C_{n-i-1}.$$

$\square$

We have found our recurrence relationship. This sequence is called the Catalan numbers, and it provides the solution to many problems.

---

**Theorem 5.2** (Catalan Numbers)

The explicit formula for the nth Catalan number is

$$\frac{1}{n+1} \binom{2n}{n}.$$

---

We won't prove this in the handout.

As previously discussed, many other problems also have the Catalan numbers as their solutions. This is because we can form a bijection between such a problem with example 4.1. One of those problems is shown below:

---

**Example 5.3**

How many ways are there to arrange $n$ open brackets "[" and $n$ closed brackets "]" such that when read left to right, the number of closed brackets is less than or equal to the number of open brackets?

---

*Solution.* We could try to form the recurrence again, but instead, we use a clever bijection to example 4.1. We place an duck on point $(0,0)$. Then, whenever we have an open bracket "[", we will move the duck one unit to the right. Whenever we have a closed bracket, we will move the duck one unit up. Since there are $n$ open brackets and $n$ closed brackets, the duck will end up on point $(n,n)$, and since the number of closed brackets is always less than or equal to the number of open brackets, the duck will never go above the line $x = y$. Thus, the solution to this problem is simply the Catalan numbers $C_n$. $\square$

## §6 Exercises

**Exercise 6.1.** How many ways are there to tile an 10 by 2 board with 1 by 2 dominoes such that each domino covers exactly two squares and no domino overlaps?

**Exercise 6.2** (AMC 12 2019)**.** How many binary sequences (sequences of 0's and 1's) of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0's, and contain no three consecutive 1's?

**Exercise 6.3** (AIME 2015)**.** There are $2^{10} = 1024$ possible 10-letter strings in which each letter is either an A or a B. Find the number of such strings that do not have more than 3 adjacent letters that are identical.

**Exercise 6.4.** Given a regular $2n$-gon, how many ways are there to pair vertices and draw line segments between those vertices such that no two line segments intersect?

**Exercise 6.5** (Charmander3333's Contributor Test)**.** Ms. Red wants to create a 10 question test for her students. Each question is multiple choice with 4 answer choices, $a, b, c$, and $d$. How many ways are there for Ms. Red to create the 10 question test if every 5 consecutive questions must contain at least one of each of the four answer choices?

**Exercise 6.6** (AIME 2001)**.** A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible?

**Exercise 6.7** (AMC 12 A 2019)**.** Let $\triangle A_0 B_0 C_0$ be a triangle whose angle measures are exactly $59.999°$, $60°$, and $60.001°$. For each positive integer $n$, define $A_n$ to be the foot of the altitude from $A_{n-1}$ to line $B_{n-1} C_{n-1}$. Likewise, define $B_n$ to be the foot of the altitude from $B_{n-1}$ to line $A_{n-1} C_{n-1}$, and $C_n$ to be the foot of the altitude from $C_{n-1}$ to line $A_{n-1} B_{n-1}$. What is the least positive integer $n$ for which $\triangle A_n B_n C_n$ is obtuse?

**Exercise 6.8** (AIME 2016)**.** The figure below shows a ring made of six small sections which you are to paint on a wall. You have four paint colors available and will paint each of the six sections a solid color. Find the number of ways you can choose to paint each of the six sections if no two adjacent section can be painted with the same color?
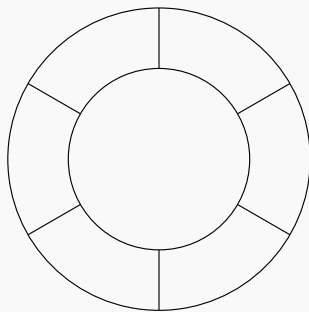
**Exercise 6.9.** Given a stairstep shape of size $n$ by $n$, where the $n = 4$ version is shown below,
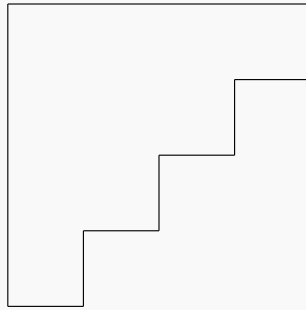


how many ways are there to tile this $n$ by $n$ stairstep shape with $n$ rectangles with integer lengths?

**Exercise 6.10** (AIME 2018)**.** Find the number of permutations of $1, 2, 3, 4, 5, 6$ such that for each $k$ with $1 \le k \le 5$, at least one of the first $k$ terms of the permutation is greater than $k$

**Exercise 6.11** (AoPS Mock Contest)**.** How many ways can an ant walk from point A to point B along the arrows using exactly 8 steps?
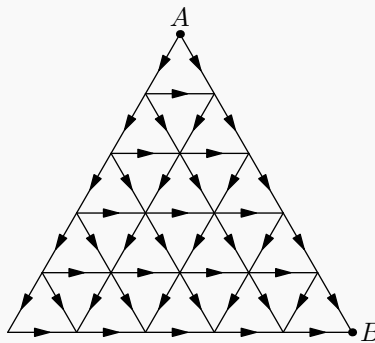


**Exercise 6.12** (Intermediate Counting and Probability)**.** Count the number of sequences of integers $a_1, a_2, a_3, \ldots, a_n$ with $a_i \le 1$ for all $i$ such that all partial sums $a_1 + a_2 + \ldots + a_i$ are non-negative.