AMC/AIME/USA(J)MO Handout

Sequences and Series in the AMC and AIME

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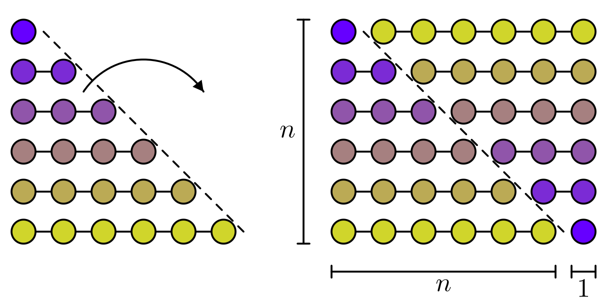

A proof without words of the sum of the first n consecutive integers.

"Everything comes to you in perfect time, space and sequence." – *Louise Hay*

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§0 Acknowledgements

This was made for the Art of Problem Solving Community out there! I would like to thank Evan Chen for his evan.sty code. In addition, all problems in the handout were likely from the AoPS Wiki.

Art of Problem Solving Community

Evan Chen's Personal Sty File

FREEMAN66's Website

NIKENISSAN's Website

And Evan says he would like this here for evan.sty:

He also helped with the hint formatting. Evan is a LaTeXgod!
And finally, please do not make any copies of this document without referencing this original one. At least cite us when you are using this document.

§1 Introduction

Sequences and series are one of the most prevalent topics in the AMC and AIME.

§2 Basics of Sequences and Series

§2.1 Definitions

Let's start by introducing a few ideas.

Definition 2.1 (Sequence) — A **sequence** is an ordered list of numbers.

Definition 2.2 (Partial Sum) — A **partial sum** is the sum of a portion of the sequence. Partial sums are sometimes called **finite series**.

Definition 2.3 (Series) — A **series** is the sum of the elements in a given sequence.

§2.2 Summation Notation

To keep things simple we introduce summation notation.

Definition 2.4 (Summation Notation) — In a summation, there is an index going from a starting point to an ending point. The given element is tested through all values from the start to the end (inclusive), and then all values of the element are summed. **Summation notation** is also sometimes called sigma notation. An example would be

$$\sum_{a=1}^{b} f(a) = f(1) + f(2) + \cdots + f(b-1) + f(b).$$

Example 2.5

Write $1 + 2 + \ldots + n$ in summation notation.

Solution. Denote x as the index of the summation. Notice the starting point of the summation is 1 and the ending point of the summation is n. Our element is x in this scenario, thus our summation can be written as

$$\boxed{\sum_{x=1}^{n} x}.$$

□

Theorem 2.6 (Distributive Property of Summations)

For all real constants k,

$$\sum k \cdot f(x) = k \sum f(x).$$

Example 2.7
Evaluate
$$\sum_{x=1}^{5} \frac{3 \cdot x!}{(x-1)!}.$$

Solution. By the Distributive Property of Summations,
$$\sum_{x=1}^{5} \frac{3 \cdot x!}{(x-1)!} = 3\sum_{x=1}^{5} \frac{x!}{(x-1)!}.$$
Notice $\frac{x!}{(x-1)!} = x$. Thus,
$$3\sum_{x=1}^{5} \frac{x!}{(x-1)!} = 3\sum_{n=1}^{5} x.$$
$\sum_{x=1}^{5} x = 1+2+3+4+5 = 15$. Therefore,
$$3\sum_{x=1}^{5} x = 3 \cdot 15 = \boxed{45}.$$

□

Theorem 2.8 (Commutative Property of Summations)
For functions $f(x)$ and $g(x)$,
$$\sum \left(f(x) \pm g(x)\right) = \sum f(x) \pm \sum g(x).$$

Summations simplify an expression a lot, and make it easier and more convenient to work with.

Exercise 2.9. Simplify
$$\sum (x+1)^2 - \sum x^2.$$

Exercise 2.10. Compute the sum
$$\sum_{k=1}^{\infty} \frac{k^2}{2^k}.$$

§3 Arithmetic Sequences and Series

Definition 3.1 (Arithmetic Sequence) — An **arithmetic sequence** is a sequence of numbers in which each term is given by adding a fixed value to the previous term.

For example, $-2, 1, 4, 7, 10, \ldots$ is an arithmetic sequence because each term is three more than the previous term. In this case, 3 is called the **common difference** of the sequence. More formally, an arithmetic sequence a_n is defined recursively by a first term a_1 and $a_n = a_{n-1} + d$ for $n \geq 2$, where d is the common difference. Explicitly:

Theorem 3.2 (Terms of an Arithmetic Sequence)
The n^{th} term in an arithmetic sequence is described

$$a_n = a_1 + d(n-1),$$

where a_n is the n^{th} term, a_1 is the first term, and d is the difference between consecutive terms.

Theorem 3.3 (Sum of an Arithmetic Sequence)
The sum of the first n terms of an arithmetic sequences is

$$s_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n-1)d).$$

Proof. We can prove this with **induction**, which will not be taught or explained in this handout. □

The following formulas are a direct consequence of the aforementioned theorems.

Corollary 3.4 (Sum of First n Positive Integers)
For all positive integers n,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Corollary 3.5 (Sum of First n Even Integers)
For all positive integers n,

$$2 + 4 + \ldots + 2n = n(n+1).$$

Corollary 3.6 (Sum of First n Odd Integers)
For all positive integers n,

$$1 + 3 + 5 + \ldots + (2n-1) = n^2.$$

Example 3.7 (AIME I 2005/2)
For each positive integer k, let S_k denote the increasing arithmetic sequence of integers whose first term is 1 and whose common difference is k. For example, S_3 is the sequence $1, 4, 7, 10, \ldots$. For how many values of k does S_k contain the term 2005?

Solution. Suppose that the nth term of the sequence S_k is 2005. Then $1 + (n-1)k = 2005 \implies k(n-1) = 2004 = 2^2 \cdot 3 \cdot 167$. Because k when multiplied by an integer is equal to 2004, in order words we can say that k is a divisor/divides 2004. By the formula for the number of divisors, there are $(2+1)(1+1)(1+1) = 12$ divisors of $2^2 \cdot 3^1 \cdot 167^1$. □

Example 3.8 (AIME II 2011/3)

The degree measures of the angles in a convex 18-sided polygon form an increasing arithmetic sequence with integer values. Find the degree measure of the smallest angle.

Solution. The average angle in an 18-gon is 160°. In this arrangement, the average is equivalent to the middle, so the center two terms of the succession average to 160°. In this manner for some certain (the grouping is expanding and hence non-consistent) whole number d, the center two terms are $(160-d)^\circ$ and $(160+d)^\circ$. Since the progression is $2d$, the last term of the arrangement is $(160+17d)^\circ$, which is under 180°, since the polygon is raised. This gives $17d < 20$, so the lone appropriate positive whole number d is 1. The initial term is then $(160-17)^\circ = \boxed{143}$. □

Example 3.9 (AIME I 2012/2)

The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence.

Solution. If the sum of the original sequence is $\sum_{i=1}^{n} a_i$ then the sum of the new sequence can be expressed as $\sum_{i=1}^{n} a_i + (2i-1) = n^2 + \sum_{i=1}^{n} a_i$. Therefore, $836 = n^2 + 715 \implies n = 11$. Now the middle term of the original sequence is simply the average of all the terms, or $\frac{715}{11} = 65$, and the first and last terms average to this middle term, so the desired sum is simply three times the middle term, or $\boxed{195}$. □

Example 3.10 (AMC 8 2015/18)

An arithmetic sequence is a sequence in which each term after the first is obtained by adding a constant to the previous term. Each row and each column in this 5×5 array is an arithmetic sequence with five terms. What is the value of X?

1				25
		X		
17				81

(A) 21 **(B)** 31 **(C)** 36 **(D)** 40 **(E)** 42

Solution. The middle term of the first row is $\frac{25+1}{2} = 13$ since the middle number is just the average in an arithmetic sequence. Similarly, the middle of the bottom row is $\frac{17+81}{2} = 49$. Applying this again for the middle column, the answer is $\frac{49+13}{2} = \boxed{31}$. □

Exercise 3.11 (AHSME 1969/33)**.** Let S_n and T_n be the respective sums of the first n terms of two arithmetic series. If $S_n : T_n = (7n+1) : (4n+27)$ for all n, the ratio of the eleventh term of the first series to the eleventh term of the second series is:

(A) $4 : 3$ (B) $3 : 2$ (C) $7 : 4$ (D) $78 : 71$ (E) undetermined

Exercise 3.12 (AIME 1999/1)**.** Find the smallest prime that is the fifth term of an increasing arithmetic sequence, all four preceding terms also being prime.

§4 Geometric Sequences and Series

Definition 4.1 (Geometric Sequence) — A **geometric sequence** is a sequence of numbers in which each term is a fixed multiple of the previous term.

For example: $1, 2, 4, 8, 16, 32, \ldots$ is a geometric sequence because each term is twice the previous term. In this case, 2 is called the common ratio of the sequence. More formally, a geometric sequence may be defined recursively by:

$$a_n = r \cdot a_{n-1}, n > 1,$$

with a fixed first term a_1 and common ratio r. Using this definition, the nth term has the closed-form:

$$a_n = a_1 \cdot r^{n-1}.$$

Theorem 4.2 (Sum of a Finite Geometric Sequence)
The sum of the first n terms of a geometric sequence is given by

$$S_n = a_1 + a_2 + \cdots + a_n = a_1 \cdot \frac{r^n - 1}{r - 1},$$

where a_1 is the first term in the sequence, and r is the common ratio.

Proof. Let

$$S_n = a_1 + a_2 + \cdots + a_n.$$

Thus, the equation can be re-written as

$$S_n = a_1 + a_1 r + \cdots + a_1 r^{n-1}.$$

Multiplying both sides by r,

$$S_n r = a_1 r + a_1 r^2 + \cdots + a_1 r^n.$$

Subtracting the original equation from this equation,

$$S_n r - S_n = a_1 r^n - a_1.$$

Factoring S_n out of the expression on the LHS and a_1 out of the expression on the RHS,

$$S_n(r-1) = a_1(r^n - 1).$$

Isolating S_n,

$$S_n = a_1 \cdot \frac{r^n - 1}{r - 1}.$$

□

Definition 4.3 (Infinite Geometric Sequence) — An **infinite** geometric sequence is a geometric sequence with an infinite number of terms.

If the absolute value of the common ratio is less than 1, the terms will approach 0 and the sum of the terms will approach a fixed limit. We say that the sum of the terms of this sequence is a convergent sum.

Theorem 4.4 (Sum of an Infinite Geometric Sequence)
The general formula for the sum of such a sequence is

$$S = \frac{a_1}{1-r}.$$

Proof. Let

$$S = a_1 + a_2 + \cdots .$$

Re-writing the equation,

$$S = a_1 + a_1 r + \cdots .$$

Multiplying both sides by r,

$$Sr = a_1 r + a_1 r^2 + \cdots .$$

Subtracting this equation from the original equation,

$$S - Sr = a_1.$$

Factoring S out of the expression on the LHS,

$$S(1-r) = a_1.$$

Isolating S,

$$S = \frac{a_1}{(1-r)}.$$

□

For instance, the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$, sums to 2.

Exercise 4.5. Find the value of

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$$

One common instance of summing infinite geometric sequences is the decimal expansion of most rational numbers. For instance, $0.33333\ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \ldots$ has first term $a_0 = \frac{3}{10}$ and common ratio $\frac{1}{10}$, so the infinite sum has value $S = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{1}{3}$, just as we would have expected.

Example 4.6 (AIME II 2002/11)
Two distinct, real, infinite geometric series each have a sum of 1 and have the same second term. The third term of one of the series is $1/8$, and the second term of both series can be written in the form $\frac{\sqrt{m}-n}{p}$, where m, n, and p are positive integers and m is not divisible by the square of any prime. Find $100m + 10n + p$.

Solution. Suppose the second term of the two distinct, real, infinite geometric series is x. Then, the first term is $8x^2$ common ratio is $\frac{1}{8x}$ for one of the series. Therefore, by the sum of an infinite geometric sequence,

$$1 = \frac{8x^2}{1 - \frac{1}{8x}}$$

$$\implies 64x^3 - 8x + 1 = 0$$

$$\implies (4x-1)(16x^2 + 4x - 1) = 0$$

$$\implies x = \frac{1}{4}, \frac{-1+\sqrt{5}}{8}, \frac{-1-\sqrt{5}}{8}.$$

Because the second term of both series is written in the form $\frac{\sqrt{m}-n}{p}$, the only applicable solution is $x = \frac{\sqrt{5}-1}{8}$. Thus, our answer is $\boxed{518}$. □

Example 4.7 (AIME II 2011/5)

The sum of the first 2011 terms of a geometric sequence is 200. The sum of the first 4022 terms is 380. Find the sum of the first 6033 terms.

Solution. The sum of the first 2011 terms is 200, second 2011 is 180, and so the ratio of the second 2011 terms to the first 2011 terms is $\frac{9}{10}$. Following the same pattern, the sum of the third 2011 terms is $\frac{9}{10} * 180 = 162$. Thus, the sum of the first 6033 terms is $200 + 180 + 162 = \boxed{542}$. □

Example 4.8 (AIME II 2012/2)

Two geometric sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ have the same common ratio, with $a_1 = 27$, $b_1 = 99$, and $a_{15} = b_{11}$. Find a_9.

Solution. Let r be the common ratio of the two geometric sequences. Then, we see that $a_1 \cdot r^{14} = b_1 \cdot r^{10} \implies r^4 = \frac{99}{27} = \frac{11}{3}$. Thus, $a_9 = a_1 \cdot r^8 = a_1 \cdot (r^4)^2 = 27 \cdot \left(\frac{11}{3}\right)^2 = 27 \cdot \frac{121}{9} = \boxed{363}$. □

Example 4.9 (AIME I 2016/1)

For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \cdots.$$

Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.

Solution. By infinite geometric series, $S(a) = \frac{12}{1-a}$ and $S(-a) = \frac{12}{1+a}$. So,

$$S(a)S(-a) = \left(\frac{12}{1-a}\right)\left(\frac{12}{1+a}\right) = \frac{144}{1-a^2} = 2016.$$

Notice

$$S(a) + S(-a) = \frac{12}{1-a} + \frac{12}{1+a} = \frac{24}{1-a^2},$$

and so the answer is $\frac{2016}{6} = \boxed{336}$. □

Exercise 4.10 (AIME II 2002/3)**.** It is given that $\log_6 a + \log_6 b + \log_6 c = 6$, where a, b, and c are positive integers that form an increasing geometric sequence and $b - a$ is the square of an integer. Find $a + b + c$.

Exercise 4.11 (AIME I 2009/1)**.** Call a 3-digit number *geometric* if it has 3 distinct digits which, when read from left to right, form a geometric sequence. Find the difference between the largest and smallest geometric numbers.

§5 Arithmetico-Geometric Sequence

Definition 5.1 (Arithmetico-Geometric Sequence and Series) — An **arithmetico-geometric series** is the sum of consecutive terms in an **arithmetico-geometric sequence** defined as:

$$t_n = a_n g_n,$$

where a_n and g_n are the nth terms of arithmetic and geometric sequences, respectively.

Example 5.2

Compute the sum

$$\sum_{k=1}^{\infty} \frac{k}{4^k}.$$

Solution. Let the sum be S. Then

$$S = \frac{1}{4} + \frac{2}{4^2} + \frac{3}{4^3} + \dots,$$

and if we divide by 4,

$$\frac{S}{4} = \frac{1}{4^2} + \frac{2}{4^3} + \dots.$$

Note that if we subtract these in a special way (by subtracting the ones with common denominators), we get

$$\frac{3S}{4} = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3},$$

so $S = \frac{4}{9}$. □

Theorem 5.3 (Sum of a Finite Arithmetico-Geometric Sequence)

The sum of the first n terms of an arithmetico-geometric sequence is

$$\frac{a_n g_{n+1}}{r-1} - \frac{x_1}{r-1} - \frac{d(g_{n+1} - g_2)}{(r-1)^2},$$

where d is the common difference of a_n and r is the common ratio of g_n. We can also write it as

$$\frac{a_n g_{n+1} - x_1 - drS_g}{r-1},$$

where S_g is the sum of the first n terms of g_n.

§6 Telescoping Series

Definition 6.1 (Telescoping Series) — A **telescoping** series is a series whose partial sums eventually only have a fixed number of terms after cancellation.

The cancellation technique, with part of each term canceling with part of the next term, is known as the method of differences.

Example 6.2

Find the value of

$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \times \frac{6}{7}.$$

Solution. Performing cancellations, we arrive at $\boxed{\frac{1}{7}}$. □

The method of splitting the fraction is known as **partial fraction decomposition**.

Theorem 6.3 (Partial Fraction Decomposition)

The method is detailed below:

- Factor the denominator
- Solve the decomposition according to the right form

A few notes on decomposing:

- Linear Factors: $\dfrac{5x+3}{(x+1)(x+4)} = \dfrac{A}{x+1} + \dfrac{B}{x+4}$
- Repeated Linear Factors: $\dfrac{2x-4}{(x-2)(x+3)^2} = \dfrac{A}{x-2} + \dfrac{B}{x+3} + \dfrac{C}{(x+3)^2}$
- Irreducible Quadratic Factors: $\dfrac{2x^2-3x-1}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}$

Exercise 6.4. Decompose

$$\frac{x^2+x-5}{(x-2)(x-1)^2}.$$

Partial Fraction Description is often able to be utilized in Telescoping.

Example 6.5

Find the value of

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots$$

Solution. If we split $\frac{1}{n(n+1)}$, we get

$$\frac{1}{n} - \frac{1}{n+1}.$$

Thus,

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots = \boxed{1}.$$

□

Example 6.6

Find the value of

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\ldots$$

Solution. Note that

$$1 - \frac{1}{n^2} = \frac{n^2 - 1}{n^2} = \frac{(n-1)(n+1)}{n^2}.$$

Thus,

$$\frac{1 \cdot 3}{2^2} \cdot \frac{2 \cdot 4}{3^2} \cdot \frac{3 \cdot 5}{4^2} \cdot \frac{4 \cdot 6}{5^2} \cdot \ldots = \frac{1}{4} \cdot 2 = \frac{1}{2}.$$

□

Example 6.7 (Stanford 2011)

Evaluate the sum

$$\sum_{n \geq 1} \frac{7n + 32}{n(n+2)} \cdot \left(\frac{3}{4}\right)^n.$$

Solution. Note that

$$\frac{7n+32}{n(n+2)} = \frac{16}{n} - \frac{9}{n+2}.$$

The sum telescopes according to the fact that $9 = (3/4)^2 \cdot 16$:

$$\sum_{n \geq 1} \frac{16}{n}\left(\frac{3}{4}\right)^n - \frac{16}{n+2}\left(\frac{3}{4}\right)^{n+2}.$$

The trailing terms in the sum tend to zero as $n \to \infty$ so the answer is $\frac{16}{1} \cdot \frac{3}{4} + \frac{16}{2} \cdot \frac{9}{16} = 12 + \frac{9}{2} = \frac{33}{2}$. □

Exercise 6.8 (USAMTS 3/4/11)**.** Determine the value of

$$S = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \cdots + \sqrt{1 + \frac{1}{1999^2} + \frac{1}{2000^2}}$$

Exercise 6.9 (AIME I 2002/4)**.** Consider the sequence defined by $a_k = \dfrac{1}{k^2 + k}$ for $k \geq 1$. Given that $a_m + a_{m+1} + \cdots + a_{n-1} = \dfrac{1}{29}$, for positive integers m and n with $m < n$, find $m + n$.

§7 Recursive Sequences

Definition 7.1 (Recursive Sequence) — A **recursive sequence** is a sequence defined upon previous terms.

Example 7.2
The most famous example is the **Fibonacci Sequence**:

$$F_n = F_{n-1} + F_{n-2},$$

where $F_0 = F_1 = 1$.

Definition 7.3 (Base Case) — A **base case** is one of the first few terms of a sequence, to give it an initial value.

For example, in the sequence $a_n = a_{n-1} + a_{n-2}$, $a_0 = a_1 = 1$ would be the *base cases*.

Definition 7.4 (Closed Form) — The **closed form**, also known as the **explicit form**, of a recursive sequence is when we do not use previous terms to define the next term.

Example 7.5
Find the closed form of

$$a_n = a_{n-1} + 1,$$

where $a_0 = 1$.

Solution. The first few terms of the sequence are

$$a_0 = 1, a_1 = 2, a_2 = 3.$$

This pattern is very obviously $a_n = n + 1$. □

Induction will be required to prove this solution holds forever, which again, will not be taught in this handout.

Example 7.6 (AMC 10A 2000/6)
The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ starts with two 1s, and each term afterward is the sum of its two predecessors. Which one of the ten digits is the last to appear in the units position of a number in the Fibonacci sequence?

(A) 0 (B) 4 (C) 6 (D) 7 (E) 9

Solution. Note that any digits other than the units digit will not affect the answer. So to make computation quicker, we can just look at the Fibonacci sequence in mod 10:

$$1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, \ldots$$

The last digit to appear in the units position of a number in the Fibonacci sequence is $6 \implies \boxed{\text{C}}$. □

Example 7.7 (AMC 12 A 2007/25)
Call a set of integers *spacy* if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, are *spacy*?

Solution. Let S_n denote the number of *spacy* subsets of $\{1, 2, 3, \ldots, n\}$. Then, $S_0 = 1, S_1 = 2, S_2 = 3$.

The *spacy* subsets of S_{n+1} either has or does not have $n + 1$. There are S_n *spacy* subsets that do not contain $n + 1$, and S_{n-2} *spacy* subsets that do contain $n + 1$ because if you take out $n + 1$ from any *spacy* subset, the max element is $n - 2$, and are possible constructions from S_{n-2}.

So, we can form the recursion

$$S_{n+1} = S_n + S_{n-2}.$$

Bashing to S_{12}, we find that the number of subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, that are *spacy* is $\boxed{129}$. □

Exercise 7.8. Derive a closed form for the Fibonacci numbers defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

Exercise 7.9 (AIME I 2006/11)**.** A collection of 8 cubes consists of one cube with edge-length k for each integer $k, 1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:

- Any cube may be the bottom cube in the tower.
- The cube immediately on top of a cube with edge-length k must have edge-length at most $k + 2$.

Let T be the number of different towers that can be constructed. What is the remainder when T is divided by 1000?

For more material on recursion, view Euclid's Orchard's Recursion in the AMC and AIME Handout and/or Primeri's Recursion Handout.

§8 Problems

Problem 8.1. Consider the arithmetic sequence with first term -500 and common difference 7. Find the 200th term in the sequence.

Problem 8.2. Consider the arithmetic sequence with first term -500 and common difference 7. Find k if $a_k = 25$.

Problem 8.3. Find the value of

$$1 + 2 + 3 + \ldots + 100.$$

Problem 8.4. Find the next term of $2, 8, 32, 128, \ldots$

Problem 8.5. What is a_7, if

$$a_n = a_{n-1} + a_{n-2},$$

and $a_0 = a_1 = 2$?

Problem 8.6 (AMC 10 B 2002/19)**.** Suppose that $\{a_n\}$ is an arithmetic sequence with

$$a_1 + a_2 + \cdots + a_{100} = 100 \text{ and } a_{101} + a_{102} + \cdots + a_{200} = 200.$$

What is the value of $a_2 - a_1$?

(A) 0.0001 (B) 0.001 (C) 0.01 (D) 0.1 (E) 1

Problem 8.7 (AMC 10 B 2004/10)**.** A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows does it contain?

(A) 5 (B) 8 (C) 9 (D) 10 (E) 11

Problem 8.8 (AMC 10 A 2006/19)**.** How many non-similar triangles have angles whose degree measures are distinct positive integers in arithmetic progression?

(A) 0 (B) 1 (C) 59 (D) 89 (E) 178

Problem 8.9 (AMC 10 B 2016/16)**.** The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

(A) $\frac{1+\sqrt{5}}{2}$ **(B)** 2 **(C)** $\sqrt{5}$ **(D)** 3 **(E)** 4

Problem 8.10 (AMC 10 A 2020/21)**.** There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \ldots < a_k$ such that

$$\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_k}.$$

What is k?

(A) 117 **(B)** 136 **(C)** 137 **(D)** 273 **(E)** 306

Problem 8.11 (AIME 1984/1)**.** Find the value of $a_2 + a_4 + a_6 + a_8 + \ldots + a_{98}$ if a_1, a_2, $a_3 \ldots$ is an arithmetic progression with common difference 1, and $a_1 + a_2 + a_3 + \ldots + a_{98} = 137$.

Problem 8.12 (AIME 1989/7)**.** If the integer k is added to each of the numbers 36, 300, and 596, one obtains the squares of three consecutive terms of an arithmetic series. Find k.

Problem 8.13 (AIME II 2004/9)**.** A sequence of positive integers with $a_1 = 1$ and $a_9 + a_{10} = 646$ is formed so that the first three terms are in geometric progression, the second, third, and fourth terms are in arithmetic progression, and, in general, for all $n \geq 1$, the terms $a_{2n-1}, a_{2n}, a_{2n+1}$ are in geometric progression, and the terms a_{2n}, a_{2n+1}, and a_{2n+2} are in arithmetic progression. Let a_n be the greatest term in this sequence that is less than 1000. Find $n + a_n$.

Problem 8.14 (AIME II 2005/3)**.** An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has 10 times the sum of the original series. The common ratio of the original series is $\frac{m}{n}$ where m and n are relatively prime integers. Find $m + n$.

Problem 8.15 (AIME II 2016/9)**.** The sequences of positive integers $1, a_2, a_3, \ldots$ and $1, b_2, b_3, \ldots$ are an increasing arithmetic sequence and an increasing geometric sequence, respectively. Let $c_n = a_n + b_n$. There is an integer k such that $c_{k-1} = 100$ and $c_{k+1} = 1000$. Find c_k.

Problem 8.16 (Math League HS 2000-2001)**.** Evaluate

$$\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)+\left(\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots\right)+\left(\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right)+\cdots.$$

More formally, what is the sum of all fractions of the form $\frac{1}{(m+1)^{(n+1)}}$, where m and n range over positive integers?

Problem 8.17 (Purple Comet HS 2004)**.** Define $a_k = (k^2+1)k!$ and $b_k = a_1 + a_2 + a_3 + \cdots + a_k$. Let

$$\frac{a_{100}}{b_{100}} = \frac{m}{n}$$

where m and n relatively prime natural numbers. Find $n-m$.

Problem 8.18 (OMO Fall 2013/24)**.** The real numbers $a_0, a_1, \dots, a_{2013}$ and $b_0, b_1, \dots, b_{2013}$ satisfy $a_n = \frac{1}{63}\sqrt{2n+2} + a_{n-1}$ and $b_n = \frac{1}{96}\sqrt{2n+2} - b_{n-1}$ for every integer $n = 1, 2, \dots, 2013$. If $a_0 = b_{2013}$ and $b_0 = a_{2013}$, compute

$$\sum_{k=1}^{2013}\left(a_k b_{k-1} - a_{k-1} b_k\right).$$

Problem 8.19 (Putnam 2013/B1)**.** For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1, c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

Problem 8.20 (AoPS)**.** If a, b, and c form an arithmetic progression, and

$$a = x^2 + xy + y^2,$$
$$b = x^2 + xz + z^2,$$
$$c = y^2 + yz + z^2,$$

where $x + y + z \neq 0$, prove that x, y, and z also form an arithmetic progression.

§A Appendix A: List of Theorems and Definitions

List of Theorems

List of Definitions