AIME/USA(J)MO Handout

Trigonometry in the AIME and USA(J)MO

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For:
AoPS

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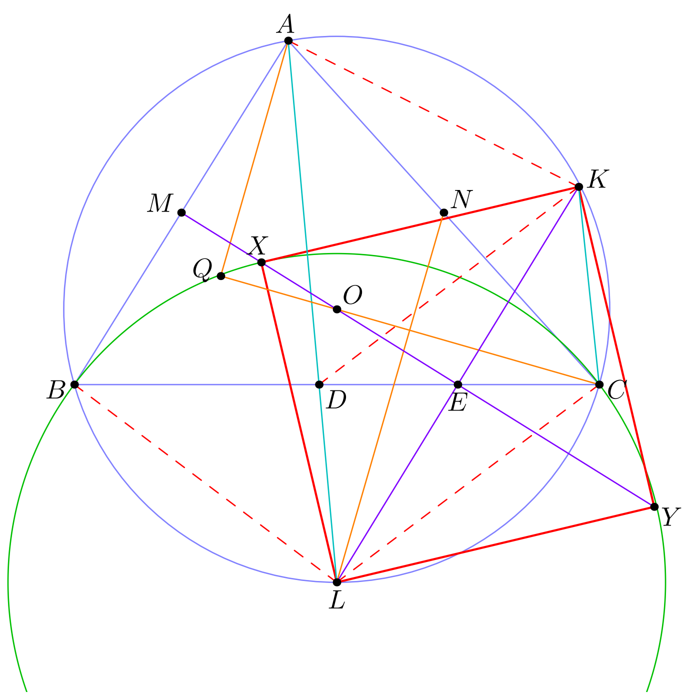


Yet another beauty by Evan. Yes, you can solve this with trigonometry.

"I was trying to unravel the complicated trigonometry of the radical thought that silence could make up the greatest lie ever told." - Pat Conroy

Contents

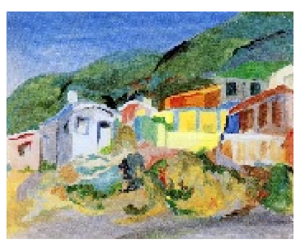
§0 Acknowledgements

This was made for the Art of Problem Solving Community out there! We would like to thank Evan Chen for his evan.sty code. In addition, all problems in the handout were either copied from the Art of Problem Solving Wiki or made by ourselves.

Art of Problem Solving Community

Evan Chen's Personal Sty File

NAMAN12's Website: Say hi!



FREEMAN66's Website: Say hi!
Note: This is a painting by Richard Feynman, who I admire a lot.

And Evan says he would like this here for evan.sty:

He also helped with the hint formatting. Evan is a LaTeXgod!
And finally, please do not make any copies of this document without referencing this original one. At least cite us when you are using this document.

§1 Introduction

§1.1 Motivation and Goals

Trigonometry is one of the main ways to solve a geometry problem. Although there are synthetic solutions, trigonometry frequently offers an solution that is very easy to find - even in the middle of the AIME or USA(J)MO. Here's a fish we will be trying to chase:

Problem 1 (2016 AIME II Problem 14)
Equilateral $\triangle ABC$ has side length 600. Points P and Q lie outside the plane of $\triangle ABC$ and are on opposite sides of the plane. Furthermore, $PA = PB = PC$, and $QA = QB = QC$, and the planes of $\triangle PAB$ and $\triangle QAB$ form a 120° dihedral angle (the angle between the two planes). There is a point O whose distance from each of A, B, C, P, and Q is d. Find d.

Geometry in three dimensions often is very hard to visualize - that is why algebraic vectors are so useful (more information in 3-D Geometry), being used as a way to easily manipulate such-things. A second such problem follows:

Problem 2 (2014 AIME II Problem 12)
Suppose that the angles of $\triangle ABC$ satisfy $\cos(3A) + \cos(3B) + \cos(3C) = 1$. Two sides of the triangle have lengths 10 and 13. There is a positive integer m so that the maximum possible length for the remaining side of $\triangle ABC$ is $\sqrt{m}$. Find m.

Note how it is impossible to solve this problem without knowledge of trigonometry - such problems will be there on the AIME. And finally, here's a third problem:

Problem 3 (2005 AIME II Problem 12)
Square $ABCD$ has center O, $AB = 900$, E and F are on AB with $AE < BF$ and E between A and $F, m\angle EOF = 45^\circ$, and $EF = 400$. Given that $BF = p + q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of any prime, find $p + q + r$.

Remark. A word of advice for those who intend to follow this document: almost all problems are from the AIME; a few HMMT and USA(J)MO problems might be scattered in, but remember we go into a fair amount of depth here. Many of the areas will have olympiad-style questions, but the underlying idea is that they could very well show up on the AIME, and most definitely olympiads.

§1.2 Contact

If do you have questions, comments, concerns, issues, or suggestions? Here are two ways to contact naman12 or freeman66:

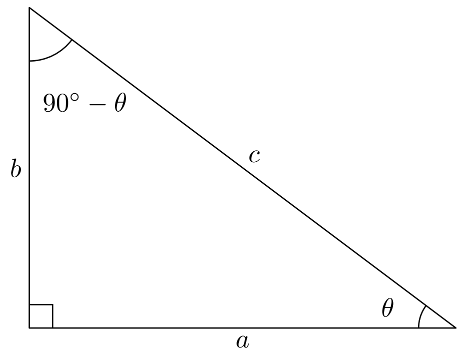
1. Send an email to realnaman12@gmail.com and I should get back to you (unless I am incorporating your suggestion into the document, then it might take a bit more time).

2. Send a private message to naman12 or freeman66 by either clicking the button that says PM or by going here and clicking New Message and typing naman12 or freeman66.

Please include something related to **Trigonometry AIME/USA(J)MO Handout** in the subject line so naman12 or freeman66 knows what you are talking about.

§2 Basic Trigonometry

We'll start out with a right triangle. It's a nice triangle - we know an angle of $90°$. What about the other angles? Let's call one θ and the other one will be $90° - \theta$:



The big question arises: how does θ even relate to a, b, c? That's why we introduce trigonometric functions. But first we need the unit circle.

§2.1 Trigonometry on the Unit Circle

Although these definitions are accurate, there is a sense in which they are lacking, because the angle θ in a right triangle can only have a measure between $0°$ and $90°$. We need a definition which will allow the domain of the sine function to be the set of all real numbers. Our definition will make use of the unit circle, $x^2 + y^2 = 1$. We first associate every real number t with a point on the unit circle. This is done by "wrapping" the real line around the circle so that the number zero on the real line gets associated with the point $(0, 1)$ on the circle. A way of describing this association is to say that for a given t, if $t > 0$ we simply start at the point $(0, 1)$ and move our pencil counterclockwise around the circle until the tip has moved t units. The point we stop at is the point associated with the number t. If $t < 0$, we do the same thing except we move clockwise. If $t = 0$, we simply put our pencil on $(0, 1)$ and don't move. Using this association, we can now define $\cos(t)$ and $\sin(t)$.

Using the above association of t with a point $(x(t), y(t))$ on the unit circle, we define $\cos(t)$ to be the function $x(t)$, and $\sin(t)$ to be the function $y(t)$, that is, we define $\cos(t)$ to be the x-coordinate of the point on the unit circle obtained in the above association, and define $\sin(t)$ to be the y coordinate of the point on the unit circle obtained in the above association.

Example 2.1

What point on the unit circle corresponds with $t = \frac{\pi}{2}$?

Solution. One loop around the circle gives an angle of 2π, implying $\frac{\pi}{2}$ is one-fourth of a loop. This brings us to the northernmost point of the circle, which gives a **vertical** component of 1 (because that is the radius of a unit circle), and a **horizontal** component of 0. This implies $\cos \frac{\pi}{2}$ is 0, and $\sin \frac{\pi}{2}$ is 1. □

Exercise 2.2. What point on the unit circle corresponds with $t = \pi$? What therefore are $\cos(\pi)$ and $\sin(\pi)$? **Hints**: 92

Exercise 2.3. What point on the unit circle correspond with $t = \frac{3\pi}{2}$? What therefore is $\cos(\frac{3\pi}{2})$? **Hints**: 72

§2.2 Definitions of Trigonometric Functions

Let us first start with a quick definition of a few important parts of a right triangle:

> **Definition 2.4** (Hypotenuse) — The **hypotenuse** of a right triangle is the side across from the right angle.

> **Definition 2.5** (Leg) — A **leg** of a right triangle is a side adjacent to the right angle and not the hypotenuse.

> **Definition 2.6** (Sine) — The **sine** of an angle θ is written as $\sin(\theta)$, and is equivalent to the ratio of the length of the side across from the angle to the length of the hypotenuse.

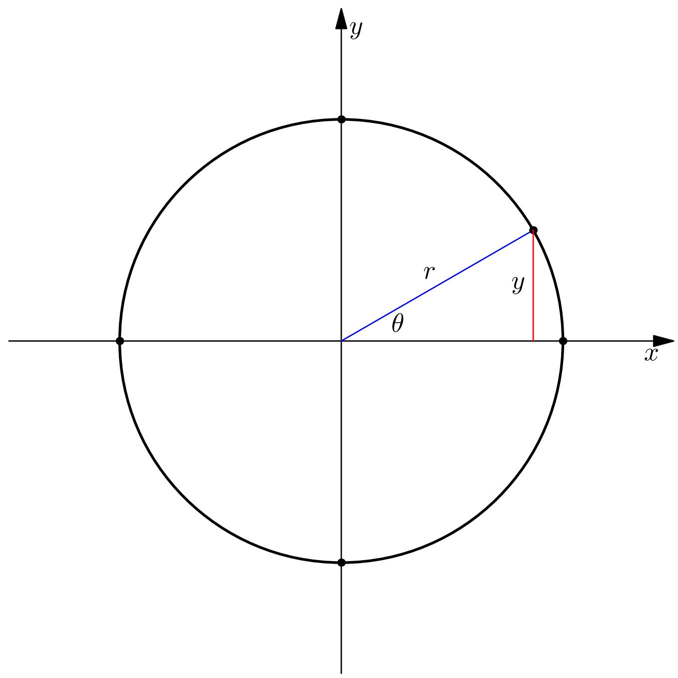


Figure 1: The length of the side opposite the angle is represented with y and the length of the hypotenuse is represented with r (which is also the radius of the circle).

Note that when this altitude to the x-axis is below the x-axis the sine of the angle is negative. When θ is between $0°$ and $180°$ or 0 rad and π rad, then $\sin(\theta)$ is positive. In addition, when θ is between $0°$ and $90°$, $\sin(\theta)$ can be viewed in the context of a right triangle as the ratio of the length side opposite the angle to the length of the hypotenuse. To see this, think about how the radius of the unit circle is the hypotenuse of the triangle in the first definition and how from there we can scale it up for larger hypotenuses without changing the value of the sine.

> **Definition 2.7** (Cosine) — The **cosine** of an angle θ is written as $\cos(\theta)$, and is equivalent to the ratio of the length of the side adjacent to the angle (not the hypotenuse) to the length of the hypotenuse.

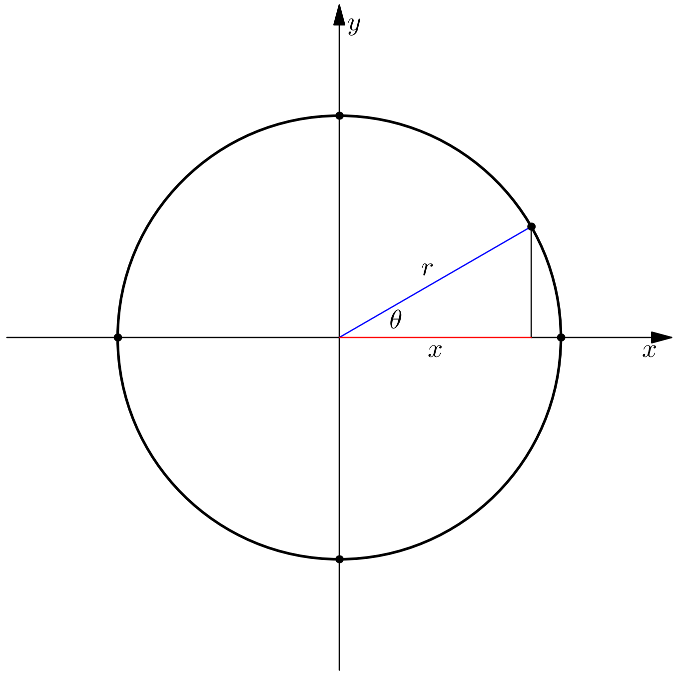
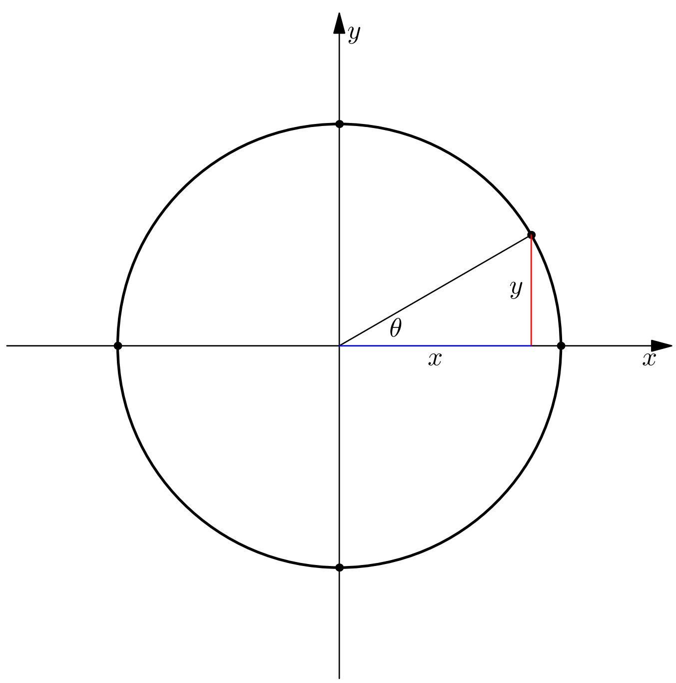


Figure 2: The length of the side adjacent to the angle is represented with x.

Similar to the sine, the cosine is negative when the point is to the left of the y-axis (i.e. for $90^\circ < \theta < 270^\circ$). In addition, for angles between 0° and 90°, the cosine can be seen in the context of a right triangle as the ratio of the lengths of the side adjacent to the angle over the hypotenuse of the triangle (again, think about scaling up the unit circle).

Definition 2.8 (Tangent) — The **tangent** of an angle θ is written as $\tan(\theta)$ and is equivalent to the ratio of the length of the line segment opposite the angle to the length of the line segment adjacent to the angle (that is not the radius of the circle, i.e. the hypotenuse).



The tangent is negative when exactly one of the sine cosine is negative. The tangent can also be seen as $\frac{\sin\theta}{\cos\theta}$. Thinking about the right triangle definitions of sine and cosine, we can get that for angles between 0° and 180°, the tangent in a right triangle is equal to the ratio of the side opposite the angle to the side adjacent to the angle.

Definition 2.9 (SOH-CAH-TOA) — If a is the length of the side opposite θ in a right triangle, and b is the length of the side adjacent to θ, and c is the length of the hypotenuse, then

$$\begin{aligned}\sin(\theta) &= \tfrac{a}{c}\\ \cos(\theta) &= \tfrac{b}{c}\\ \tan(\theta) &= \tfrac{a}{b}\\ \cot(\theta) &= \tfrac{b}{a}\\ \sec(\theta) &= \tfrac{c}{b}\\ \csc(\theta) &= \tfrac{c}{a}.\end{aligned}$$

This is commonly memorized as SOH-CAH-TOA, where S represents sine, C represents cosine, T represents tangent, all Os represent opposite (the leg opposite the angle), all As represent adjacent (the leg adjacent/-touching the angle), and H represents hypotenuse. Using the above definition of $\sin(\theta)$ and $\cos(\theta)$, we can similarly define

$$\begin{aligned}\tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)}\\ \cot(\theta) &= \frac{\cos(\theta)}{\sin(\theta)}\\ \sec(\theta) &= \frac{1}{\cos(\theta)}\\ \csc(\theta) &= \frac{1}{\sin(\theta)}\end{aligned}$$

§2.3 Radian Measure

Definition 2.10 — [Radian] A **radian** is defined to be the measure of an angle in a unit circle with arc length one.

Thus, a 90° angle corresponds to an angle of radian measure $\frac{\pi}{2}$, since the distance one fourth of the way around the unit circle is $\frac{\pi}{2}$.

It is also useful to note that an angle of measure 1° corresponds with an angle of radian measure $\frac{\pi}{180}$, since 90 of these would correspond to a right angle. Also, an angle of radian measure 1 would correspond to an angle of measure $\left(\frac{180}{\pi}\right)^\circ$, since $\frac{\pi}{2}$ of these would correspond to a right angle. These facts are enough to help you convert from degrees to radians and back, when necessary.

Exercise 2.11. What is the degree measure of the angle $\theta = \frac{\pi}{6}$? **Hints**: 66

Exercise 2.12. What is the radian measure of the angle 225°? **Hints**: 100

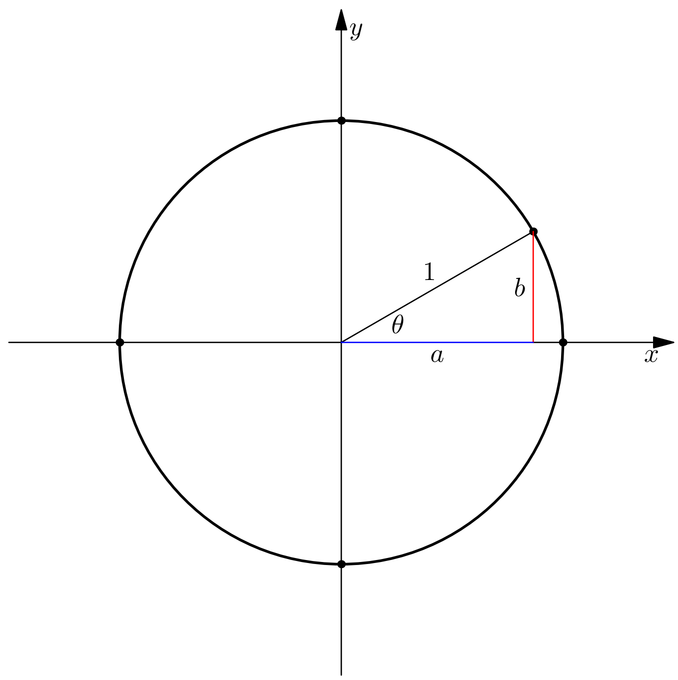
§2.4 Properties of Trigonometric Functions

Theorem 2.13 (Trigonometric Properties)
The following are some properties of functions:

1. Range of $\sin(x)$ and $\cos(x)$: $-1 \le \sin(x) \le 1$, $-1 \le \cos(x) \le 1$.
2. $\cos(x)$ is Even: $\cos(-x) = \cos(x)$.
3. $\sin(x)$ is Odd: $\sin(-x) = -\sin(x)$.
4. Periodicity: $\sin(x + 2\pi) = \sin(x)$, $\cos(x + 2\pi) = \cos(x)$.

Remark 2.14. Don't get fooled! $\sin^2(x)$ doesn't mean $\sin(\sin(x))$ – rather, it means $(\sin(x))^2$. But later, you will learn that $\sin^{-1}(x) \neq \frac{1}{\sin(x)}$ - it's actually the angle y such that $\sin(y) = x$. While this seems confusing for now, you will get accustomed to it.

Proofs. 1. Take a look at the unit circle again:



We can see that a and b are fully contained inside the unit circle. However, this means that $|a|$ and $|b|$ are at most 1 (as they are contained in a circle radius 1). Thus, we get that

$$|x| \le 1 \implies -1 \le a \le 1,$$

$$|y| \le 1 \implies -1 \le b \le 1.$$

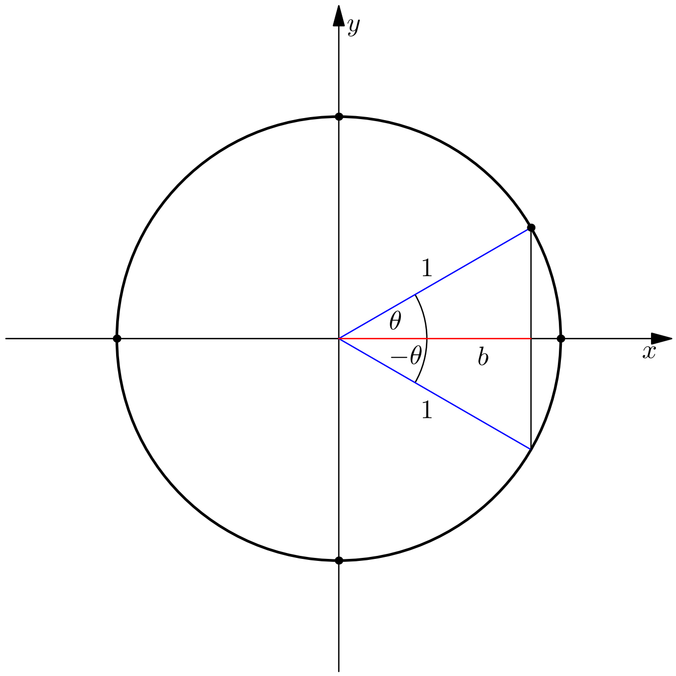
However, we know that $a = \sin x$ and $b = \cos x$, so then we get

$$-1 \le \sin x \le 1,$$

$$-1 \le \cos x \le 1.$$

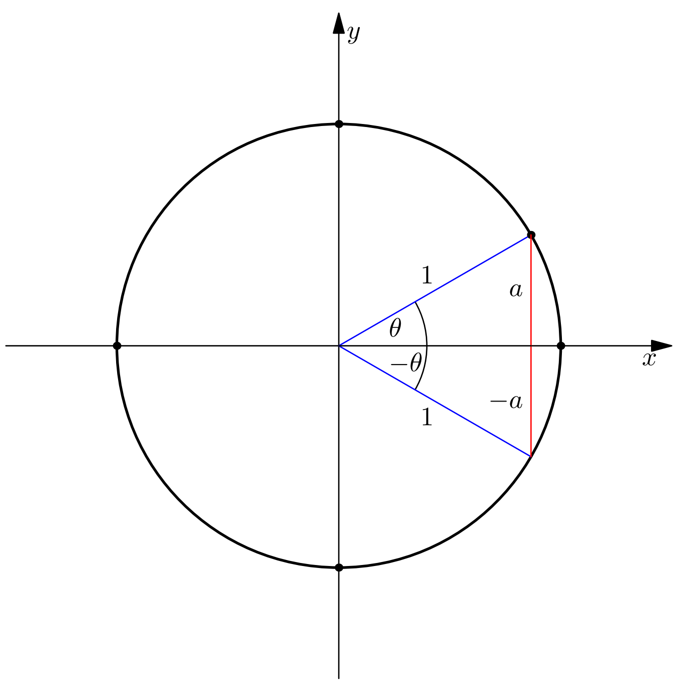
Remark 2.15. Typically, when it is unambiguous, we will resort to writing $\sin x$ instead of $\sin(x)$. However, if there is a chance of misinterpretation, we shall include parentheses.

2. Once again, we resort to the unit circle:



We see this is just a reflection over the x-axis - in particular, the value of the x-coordinate, b, stays the same. However, we know that this particular value is $\cos\theta$, so we get that $\cos\theta = \cos -\theta = b$.

3. Can you guess what we will use? The unit circle:



We see this is just a reflection over the x-axis - in particular, the value of the y-coordinate, a, becomes negative. However, we know that this particular value is $\sin\theta$, so we get that $\sin\theta = -\sin -\theta = a$.

4. Think of this visually - as $2\pi = 360^\circ$, in reality, we are just going all the way around the circle, so indeed the point corresponding to $(\cos x, \sin x)$ also corresponds to $(\cos(2\pi + x), \sin(2\pi + x))$.

□

§2.5 Graphs of Trigonometric Functions

§2.5.1 Graph of sin(x) and cos(x)

Note that from the definition of sine and cosine, it is clear that the domain of each of these is the set of all real numbers. Also, from the properties above, we know that the range of both of these is the set of numbers between -1 and 1, and that the functions are periodic. This information, together with a few points plotted as a guide, are enough to graph the two functions. Note that if we shift the graph of the sine function by $\frac{\pi}{2}$ units to the left, we get the graph of the cosine function. This is related to the fact that $\sin(x - \frac{\pi}{2}) = \cos(x)$, which we will explore more later.

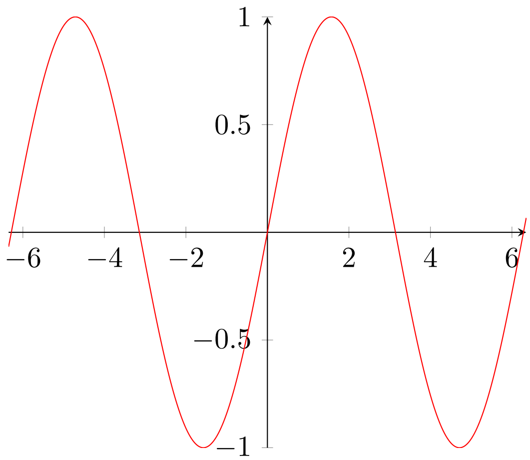


Figure 3: Graph of $\sin x$

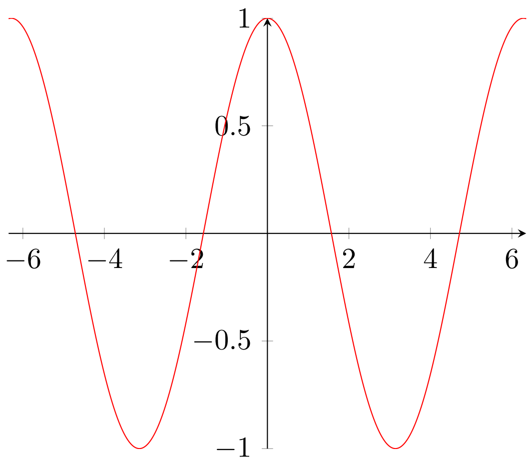


Figure 4: Graph of $\cos x$

§2.5.2 Graph of tan(x) and cot(x)

Note that the domain of $\tan(x)$ is the set of all real numbers except those at which $\cos(x) = 0$. Thus, the points $\frac{\pi}{2}$, $\frac{3\pi}{2}$, and so on aren't in the domain of $\tan(x)$. An easy way to characterize these points is to say that these are all the points which have the form $\frac{\pi}{2} + k\pi$, where k is any integer. Thus the domain of the tangent function is everything unless $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$.

Exercise 2.16. What is the domain of $\cot(x)$? **Hints**: 69

We can get a good grasp on the graph of $\tan(x)$ by plotting a few points and doing a careful analysis of the limiting behavior when x is near $\frac{\pi}{2}$ and the other points that aren't in the domain. Note that when x is a little less than $\frac{\pi}{2}$, $\sin(x)$ is close to 1, while $\cos(x)$ is close to zero (but is positive.)

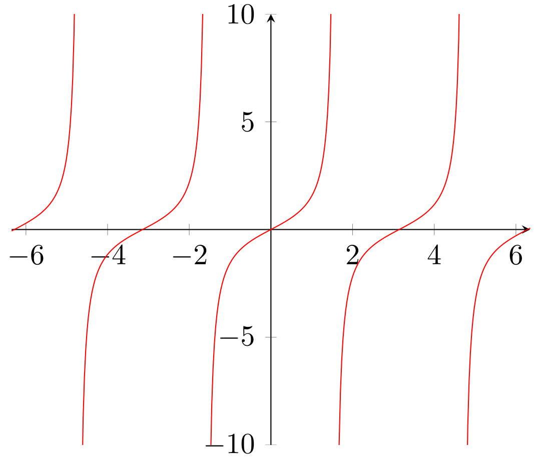


Figure 5: Graph of $\tan x$

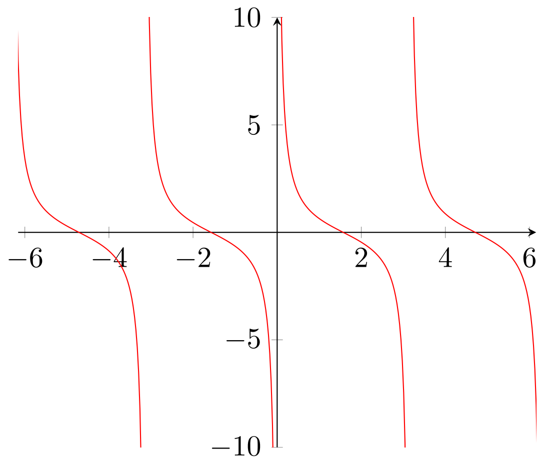


Figure 6: Graph of $\cot x$

§2.5.3 Graph of sec(x) and csc(x)

Like the tangent function, the domain of the secant function is the set of all real numbers except those which make $\cos(x)$ equal to zero. Thus the domain of the secant function is the same as the domain of the tangent function. Also, the fact that the cosine function always has values between -1 and 1 tells us that $\sec(x) = \frac{1}{\cos(x)}$ always has values less than or equal to -1 or greater than or equal 1. An analysis of the limiting behavior of $\sec(x)$ near $x = \frac{\pi}{2}$ and $\frac{-\pi}{2}$ and a few strategically plotted points leads to the graph of $y = \sec(x)$.

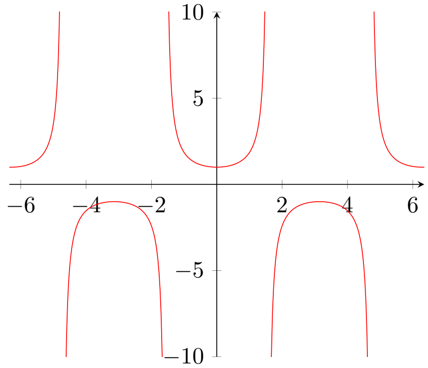


Figure 7: Graph of $\sec x$

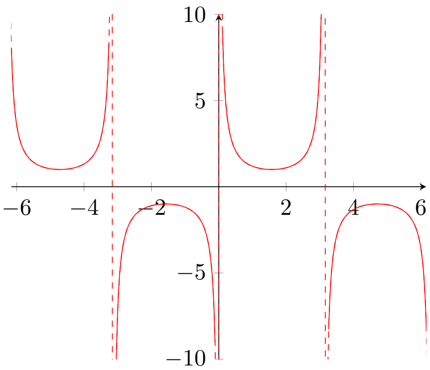


Figure 8: Graph of $\csc x$

§2.5.4 Notes on Graphing

These notes were contributed by AoPS User AOPS12345678910.

1. The **amplitude** of a graph that models a tangent equation f (i.e. $f(x) = a\tan(bx + c) + d$) is equivalent to $f\left(\frac{\pi}{4}\right)$.

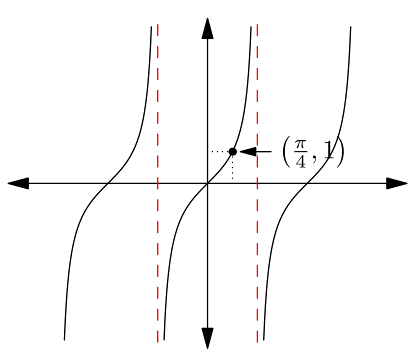


Figure 9: Graph of $\tan x$. 1 is the amplitude in this case.

2. When graphing $\sec x$, it helps to first sketch $\cos x$. Similarly, when graphing $\csc x$, it helps to first sketch $\sin x$.

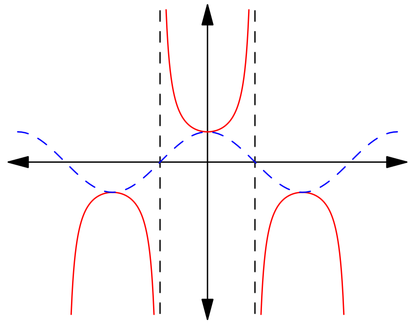


Figure 10: Graph of $\sec x$, with $\cos x$ in dashed blue.

§2.6 Bounding Sine and Cosine

The following theorem is extremely trivial but extremely useful. It is analogous to the "Trivial Inequality" of trigonometry:

Theorem 2.17 (Bounds of $\sin\theta$ and $\cos\theta$)
For all angles θ,

$$-1 \le \sin\theta \le 1,$$

$$-1 \le \cos\theta \le 1.$$

Remark 2.18. The angle θ is actually a Greek Letter, theta, and is typically used to represents angles.

Proof. Refer to Property 3 of Trigonometric Properties. □

Exercise 2.19. Bound $\tan\theta, \cot\theta, \sec\theta$, and $\csc\theta$. **Hints**: 49 95

Exercise 2.20 (AIME 1991/4)**.** How many real numbers x satisfy the equation $\frac{1}{5}\log_2 x = \sin(5\pi x)$? **Hints**: 18 19

§2.7 Periodicity

From the graphs of $\sin x$ and $\cos x$, one intuitively knows sine and cosine have periods.

Theorem 2.21 (Periods of Trigonometric Functions)

The periods of the following functions are:

1. **sine**: 2π
2. **cosine**: 2π
3. **tangent**: π
4. **cotangent**: π
5. **secant**: 2π
6. **cosecant**: 2π

Notice that both of tan and cot actually have a period of π. That's because (from the graphs) we have $\sin(x+\pi) = -\sin x$ and $\cos(x+\pi) = -\cos x$. Later, we'll also see another way to prove it with algebra.

§2.8 Trigonometric Identities

Let me now list them out:

Theorem 2.22 (Even-Odd Identities)

For all angles θ,

- $\sin(-\theta) = -\sin(\theta)$
- $\cos(-\theta) = \cos(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

Sketch of Proof. We've already seen the proof of the sin and cos. Now, the rest follows by expressing each function in terms of sin and cos. For example,

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

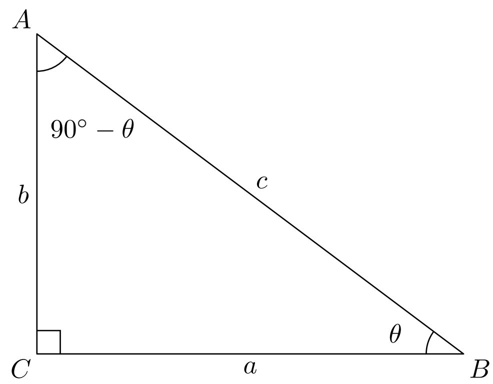
□

Theorem 2.23 (Pythagorean Identities)

For all angles θ,

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \cot^2\theta = \csc^2\theta$
- $\tan^2\theta + 1 = \sec^2\theta$

Proof. We consider the triangle $\triangle ABC$:



The Pythagorean Theorem tells us that

$$a^2 + b^2 = c^2$$

or upon dividing by c^2,

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

We now can use SOH-CAH-TOA. This tells us $\sin\theta = \frac{b}{c}$ and $\cos\theta = \frac{a}{c}$, so we can substitute to get

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

We just use the definition of Tangent and Secant:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

Now, we get

$$1 + \tan^2(\theta) = 1 + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{\sin^2(\theta) + \cos^2(\theta)}{\cos^2(\theta)}$$

However, by the first identity, we have that $\sin^2(\theta) + \cos^2(\theta) = 1$. Thus, we get

$$1 + \tan^2(\theta) = \frac{\sin^2(\theta) + \cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} = \sec^2(\theta)$$

The other one follows similarly. The definitions of Cotangent and Cosecant are:

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

Now, we get

$$1 + \cot^2(\theta) = 1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{\sin^2(\theta) + \cos^2(\theta)}{\sin^2(\theta)}$$

However, by the first identity, we have that $\sin^2(\theta) + \cos^2(\theta) = 1$. Thus, we get

$$1 + \cot^2(\theta) = \frac{\sin^2(\theta) + \cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)} = \csc^2(\theta)$$

□

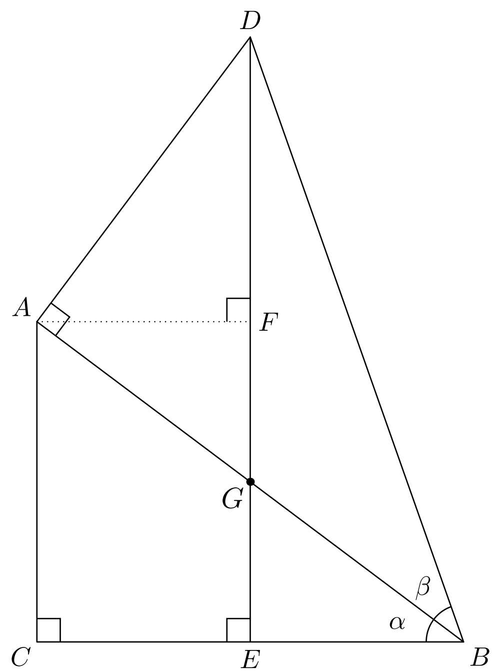
Exercise 2.24 (AIME 1995/7)**.** Given that $(1+\sin t)(1+\cos t) = \frac{5}{4}$, compute $(1-\sin t)(1-\cos t)$. **Hints**: 61

Exercise 2.25. If $\cos x + \sin x = 0.2$, compute $\cos^4 x + \sin^4 x$. **Hints**: 60

Theorem 2.26 (Addition-Subtraction Identities)
For all angles α and β,

- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$
- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
- $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

Proof. The proof of these will feel pretty magical. That's completely intended:



We'll use the above diagram to find our values. We let $DB = 1$. Then, we first note that

$$AD = \sin\beta,$$

$$AB = \cos\beta,$$

from right triangle ADB. Now, from right triangle ABC, we get

$$AC = AB\sin\alpha = \cos\beta\sin\alpha.$$

Now, we get that $AFEC$ is a rectangle, so we must have that $FE = AC = \cos\beta\sin\alpha$. Furthermore, we have that $AF \parallel BC$, so thus $\angle FAG = \angle GBE = \alpha$. Thus, we must have that

$$\angle DAF = 90^\circ - \angle GAF = 90^\circ - \alpha,$$

so

$$\angle FDA = 90^\circ - \angle DAF = \alpha.$$

Now, we can use trigonometry on the right triangle $\triangle DAF$ to get

$$DF = AD\cos\alpha = \sin\beta\cos\alpha.$$

Thus, we get from trigonometry on right triangle $\triangle BDE$

$$\sin(\alpha+\beta) = DE = DF + FE = \sin\alpha\cos\beta + \sin\beta\cos\alpha.$$

Doing it for cos and tan are essentially the same and left as an exercise. Furthermore, an additional comment is that to achieve the $\pm$ result, use the Even-Odd Identities. □

Exercise 2.27. Verify $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. **Hints**: 26

Exercise 2.28. Verify $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$. **Hints**: 86

If we let $\alpha = \beta$, then

Theorem 2.29 (Double Angle Identities)

For all angles α,

- $\sin 2\alpha = 2\sin\alpha\cos\alpha$
- $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$
- $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$
- $\csc(2\alpha) = \frac{\csc(\alpha)\sec(\alpha)}{2}$
- $\sec(2\alpha) = \frac{1}{2\cos^2(\alpha)-1} = \frac{1}{\cos^2(\alpha)-\sin^2(\alpha)} = \frac{1}{1-2\sin^2(\alpha)}$
- $\cot(2\alpha) = \frac{1-\tan^2(\alpha)}{2\tan(\alpha)}$

Exercise 2.30. Verify all the Double Angle Identities. **Hints**: 23

Exercise 2.31. The angle θ has the property that

$$\sin\theta + \cos\theta = \frac{2}{3}.$$

Compute $\sin 2\theta$. **Hints**: 70 90

Exercise 2.32. Determine all real $0 \le \theta < 2\pi$ such tath

$$1 + \sin 2\theta = \sin\left(\theta + \frac{\pi}{4}\right).$$

Hints: 93 88

Theorem 2.33 (Half Angle Identities)

For all angles θ,

- $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$
- $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$
- $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$

Make sure to understand why we have the $\pm$. We note that for any angle θ, $\cos 2\theta = \cos(2\pi + 2\theta) = \cos 2(\theta+\pi)$. However, we have that $\cos(\pi+\theta) = -\cos\theta$, and $\sin(\pi+\theta) = -\sin\theta$, so we must have the $\pm$. These aren't very hard to show - they're a direct application of the Double Angle Identities - try it as an exercise.

Exercise 2.34. Verify all the Half Angle Identities. **Hints**: 54

Theorem 2.35 (Sum to Product Identities)

For all angles θ and γ,

- $\sin\theta + \sin\gamma = 2\sin\dfrac{\theta+\gamma}{2}\cos\dfrac{\theta-\gamma}{2}$
- $\sin\theta - \sin\gamma = 2\sin\dfrac{\theta-\gamma}{2}\cos\dfrac{\theta+\gamma}{2}$
- $\cos\theta + \cos\gamma = 2\cos\dfrac{\theta+\gamma}{2}\cos\dfrac{\theta-\gamma}{2}$
- $\cos\theta - \cos\gamma = -2\sin\dfrac{\theta+\gamma}{2}\sin\dfrac{\theta-\gamma}{2}$

Proof. Let $\alpha = \frac{\theta+\gamma}{2}$ and $\beta = \frac{\theta-\gamma}{2}$. Then, we get

$$\alpha + \beta = \theta$$

$$\alpha - \beta = \gamma$$

so thus we can use Addition-Subtraction Identities to get

$$\sin\theta + \sin\gamma = \sin(\alpha+\beta) + \sin(\alpha-\beta) = (\sin\alpha\cos\beta + \sin\beta\cos\alpha) + (\sin\alpha\cos\beta - \sin\beta\cos\alpha) = 2\sin\alpha\cos\beta =$$

and looking back at our definition of α, β, we get the first of the Sum to Product Identities. The rest follow essentially the same proof and will be left as an exercise. □

Another remark - the product-to-sum identities turn out to be extremely helpful when they slap a bunch of trigonometric functions at you:

Exercise 2.36. Verify the rest of the Sum to Product Identities.

Exercise 2.37 (ARML 1988)**.** If $0^\circ < x < 180^\circ$ and $\cos x + \sin x = \frac{1}{2}$, then find (p, q) such that $\tan x = -\frac{p+\sqrt{q}}{3}$. **Hints**: 44

Exercise 2.38 (ARML)**.** Compute $\frac{\sin 13^\circ + \sin 47^\circ + \sin 73^\circ + \sin 107^\circ}{\cos 17^\circ}$. **Hints**: 35

Exercise 2.39 (AIME I 2006/12)**.** Find the sum of the values of x such that $\cos^3 3x + \cos^3 5x = 8\cos^3 4x \cos^3 x$, where x is measured in degrees and $100 < x < 200$. **Hints**: 47 22

Theorem 2.40 (Potpourri)

Some other identities:

1. $\sin(90 - \theta) = \cos(\theta)$
2. $\cos(90 - \theta) = \sin(\theta)$
3. $\tan(90 - \theta) = \cot(\theta)$
4. $\sin(180 - \theta) = \sin(\theta)$
5. $\cos(180 - \theta) = -\cos(\theta)$
6. $\tan(180 - \theta) = -\tan(\theta)$
7. $(\tan\theta + \sec\theta)^2 = \frac{1+\sin\theta}{1-\sin\theta}$
8. $\sin(\theta) = \cos(\theta)\tan(\theta)$
9. $\cos(\theta) = \frac{\sin(\theta)}{\tan(\theta)}$
10. $\sec(\theta) = \frac{\tan(\theta)}{\sin(\theta)}$
11. $\arctan(x) + \arctan(y) = \arctan\left(\dfrac{x+y}{1-xy}\right)$
12. $\sin^2(\theta) + \cos^2(\theta) + \tan^2(\theta) = \sec^2(\theta)$
13. $\sin^2(\theta) + \cos^2(\theta) + \cot^2(\theta) = \csc^2(\theta)$

Most of these can be proved by Addition-Subtraction Identities, with a few of them following from Pythagorean Identities.

Exercise 2.41. Verify the identites given in the Potpourri.

Exercise 2.42. Compute the exact numerical value of

$$\cos\frac{\pi}{9}\cos\frac{3\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}.$$

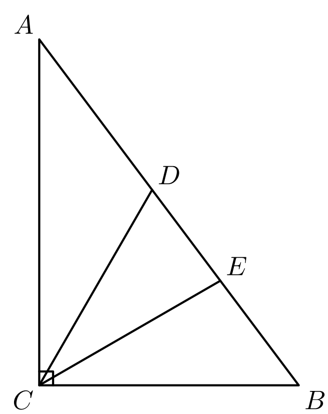
Hints: 12 97

Exercise 2.43. Compute $\sin 18^\circ$. **Hints**: 94 55

Exercise 2.44. Determine the sum of the values of $\tan\theta$ for which $0 \le \theta < \pi$ and $1 = 2004\cos\theta\cdot(\sin\theta - \cos\theta)$. **Hints**: 46

These are oftentimes very useful, as we shall see in the following examples.

Example 2.45 (AIME I 2012/12)
Let $\triangle ABC$ be a right triangle with right angle at C. Let D and E be points on $\overline{AB}$ with D between A and E such that $\overline{CD}$ and $\overline{CE}$ trisect $\angle C$. If $\frac{DE}{BE} = \frac{8}{15}$, then $\tan B$ can be written as $\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers, and p is a positive integer not divisible by the square of any prime. Find $m+n+p$.



Solution. Let $CB = 1$, and let the feet of the altitudes from D and E to $\overline{CB}$ be D' and E', respectively. Also, let $DE = 8k$ and $EB = 15k$. We see that $BD' = 15k\cos B$ and $BE' = 23k\cos B$ by right triangles $\triangle BDD'$ and $\triangle BEE'$. From this we have that $D'E' = 8k\cos B$. With the same triangles we have $DD' = 23k\sin B$ and $EE' = 15k\sin B$. From $30^\circ - 60^\circ - 90^\circ$ triangles $\triangle CDD'$ and $\triangle CEE'$, we see that $CD' = \frac{23k\sqrt{3}\sin B}{3}$ and $CE' = 15k\sqrt{3}\sin B$, so $D'E' = \frac{22k\sqrt{3}\sin B}{3}$. From our two values of $D'E'$ we get:

$$8k\cos B = \frac{22k\sqrt{3}\sin B}{3},$$

$$\frac{\sin B}{\cos B} = \frac{8k}{\frac{22k\sqrt{3}}{3}} = \tan B,$$

$$\tan B = \frac{8}{\frac{22\sqrt{3}}{3}} = \frac{24}{22\sqrt{3}} = \frac{8\sqrt{3}}{22} = \frac{4\sqrt{3}}{11}.$$

Thus, $m = 4, n = 3, p = 11$, so $4 + 3 + 11 = \boxed{018}$. □

That was a geometric problem. We'll leave you with this problem, which is algebraic:

Exercise 2.46 (AIME II 2000/15)**.** Find the least positive integer n such that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}.$$

Hints: 32 65 39

§3 Applications to Complex Numbers

Definition 3.1 (Complex Numbers) — A complex number is of the form $z = a + bi$ where a, b are real and $i = \sqrt{-1}$ is the imaginary unit. It has a conjugate $\overline{z} = a - bi$. Furthermore, it has magnitude $|z| = \sqrt{a^2 + b^2}$.

Theorem 3.2 (Complex Number Multiplication and Addition)
We multiply complex numbers $w = a + bi$ and $z = c + di$ as

$$wz = (ac - bd) + (ad + bc)i$$

and add them as

$$w + z = (a + c) + (b + d)i$$

Proof. The second follows by the associative law of addition, while the first follows by using FOIL (First Inner Outer Last) as well as the fact that $i^2 = -1$. □

Exercise 3.3 (AIME 1985/3)**.** Find c if a, b, and c are positive integers which satisfy $c = (a + bi)^3 - 107i$, where $i^2 = -1$. **Hints**: 63

Exercise 3.4 (AIME 1988/11)**.** Let $w_1, w_2, \ldots, w_n$ be complex numbers. A line L in the complex plane is called a mean line for the points $w_1, w_2, \ldots, w_n$ if L contains points (complex numbers) $z_1, z_2, \ldots, z_n$ such that

$$\sum_{k=1}^{n} (z_k - w_k) = 0.$$

For the numbers $w_1 = 32 + 170i$, $w_2 = -7 + 64i$, $w_3 = -9 + 200i$, $w_4 = 1 + 27i$, and $w_5 = -14 + 43i$, there is a unique mean line with y-intercept 3. Find the slope of this mean line. **Hints**: 78

Exercise 3.5 (AIME I 2009/2)**.** There is a complex number z with imaginary part 164 and a positive integer n such that

$$\frac{z}{z + n} = 4i.$$

Find n. **Hints**: 37

Theorem 3.6 (Euler's Theorem)
For all angles θ,
$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This is a very deep result that Euler proved from the Taylor Series (This is calculus - don't worry) of $e^x, \sin x$, and $\cos x$. But let's see some properties

Definition 3.7 (Polar Complex Numbers) — Every complex number can be expressed as $z = re^{i\theta}$ for $r = |z|$.

Theorem 3.8 (Properties of Complex Numbers)
$$\operatorname{cis}\theta_1 \cdot \operatorname{cis}\theta_2 = \operatorname{cis}(\theta_1 + \theta_2).$$

Proof. Let complex numbers $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$. Then
$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}.$$
This directly implies
$$\operatorname{cis}\theta_1 \cdot \operatorname{cis}\theta_2 = \operatorname{cis}(\theta_1 + \theta_2).$$
□

The following are other useful properties:

Exercise 3.9. Show that $\overline{w+z} = \overline{w} + \overline{z}$. **Hints**: 79

Exercise 3.10. Show that $\overline{w \cdot z} = \overline{w} \cdot \overline{z}$. **Hints**: 74

Exercise 3.11. Show that $|wz| = |w||z|$. **Hints**: 59

Theorem 3.12 (De Moivre's Theorem)
Let θ be an angle. Then
$$(\operatorname{cis}\theta)^n = \operatorname{cis}(n\theta).$$

Proof. Use Properties of Complex Numbers n times, all on the angle θ. □

Theorem 3.13 (Complex Form of Trigonometric Functions)
For some angle θ and constant k, defining $z = \operatorname{cis}\theta$,
$$\cos k\theta = \frac{1}{2}\left(z^k + \frac{1}{z^k}\right),$$
and
$$\sin k\theta = \frac{1}{2i}\left(z^k - \frac{1}{z^k}\right).$$

Proof. From De Moivre's Theorem, we get that if

$$\cos k\theta + i\sin k\theta = (\operatorname{cis}\theta)^k = z^k$$

$$\cos k\theta - i\sin k\theta = \cos -k\theta + i\sin -k\theta = (\operatorname{cis}\theta)^{-k} = z^{-k}$$

Adding these and dividing by 2 gives the first result. Subtracting these and dividing by $2i$ gives the second result. □

§3.1 Roots of Unity

Definition 3.14 (Root of Unity) — A **root of unity** is a root of the equation

$$\omega^n = 1.$$

We define ω_k as the kth root of unity, ordered by their angle with respect to the positive x-axis counter-clockwise.

These actually turn out to form a regualr polygon, as implied by the next theorem:

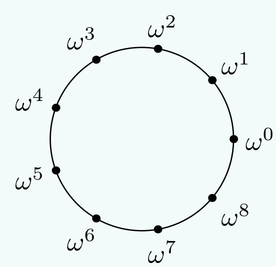
Theorem 3.15 (Roots of Unity)
Let ω be a solution to the equation

$$\omega^n = 1.$$

Then

$$\omega = e^{\frac{2k\pi i}{n}},$$

where $k = 0, 1, 2, \ldots, n-1$. This of course implies there exist n solutions to this equation (which should be intuitive from the Fundamental Theorem of Algebra).



Proof. This is more or less a direct consequence of De Moivre's Theorem. We must have that if $\omega = \operatorname{cis}\theta$, then

$$\omega^n = (\operatorname{cis}\theta)^n = \operatorname{cis} n\theta = \operatorname{cis} 2k\pi$$

so in particular, we have that $\theta = \frac{2\pi k}{n}$. Now, note by taking $1 \le k \le n$, we get n such distinct solutions. We can't have any more by the Fundamental Theorem of Algebra (see my Polynomials in the AIME Handout and Appendix A). □

Exercise 3.16 (AIME I 2004/13)**.** The polynomial $P(x) = (1 + x + x^2 + \cdots + x^{17})^2 - x^{17}$ has 34 complex roots of the form $z_k = r_k[\cos(2\pi a_k) + i\sin(2\pi a_k)], k = 1, 2, 3, \ldots, 34$, with $0 < a_1 \le a_2 \le a_3 \le \cdots \le a_{34} < 1$ and $r_k > 0$. Given that $a_1 + a_2 + a_3 + a_4 + a_5 = m/n$, where m and n are relatively prime positive integers, find $m + n$. **Hints**: 2

Exercise 3.17 (AIME 1990/10)**.** The sets $A = \{z : z^{18} = 1\}$ and $B = \{w : w^{48} = 1\}$ are both sets of complex roots of unity. The set $C = \{zw : z \in A \text{ and } w \in B\}$ is also a set of complex roots of unity. How many distinct elements are in C? **Hints**: 102

Exercise 3.18 (AIME I 2002/15)**.** Let $P(x) = x+2x^2+3x^3 \ldots 24x^{24}+23x^{25}+22x^{26} \ldots x^{47}$. Let $z_1, z_2, \ldots, z_r$ be the distinct zeros of $P(x)$, and let $z_k^2 = a_k + b_k i$ for $k = 1, 2, \ldots, r$, where a_k and b_k are real numbers. Let

$$\sum_{k=1}^{r} |b_k| = m + n\sqrt{p},$$

where m, n, and p are integers and p is not divisible by the square of any prime. Find $m+n+p$. **Hints**: 82 21

Theorem 3.19 (Vieta's Formulas in Roots of Unity)
Let the n nth roots of unity be $\omega_0, \omega_1, \ldots, \omega_n$. Then

$$\sum_{k=0}^{n-1} \mathrm{Re}(\omega_k) = \sum_{k=0}^{n-1} \cos\left(\theta_0 + \frac{2k\pi}{n}\right) = 0,$$

$$\sum_{k=0}^{n-1} \mathrm{Im}(\omega_k) = \sum_{k=0}^{n-1} \sin\left(\theta_0 + \frac{2k\pi}{n}\right) = 0.$$

$$\prod_{k=0}^{n-1} \omega_k = (-1)^{n+1}$$

Proof. Note that the ω_k are the roots of the polynomial

$$z^n - 1 = 0$$

so by Vieta's Formulas (see my Polynomials in the AIME Handout and Chapter 3), we have that the last one immediately follows and

$$\sum_{k=0}^{n-1} \omega_k = 0.$$

To be formal, I would say that 1 and i are linearly independent over $\mathbb{R}$, which means that we can seperate out into real and imaginary parts. For those of you who aren't linear algebra experts, it basically means that we can't cancel out 1 and i when adding them together, which gives us the first two. □

Theorem 3.20 (Complex Trigonometric Products)

$$\prod_{k=1}^{n-1} (1 - \omega_k)(1 + \overline{\omega_k}) = (-2i)^{n-1} \prod_{k=1}^{n-1} \sin\theta_k.$$

Proof. We note that for all $z = re^{i\theta}$,

$$z\overline{z} = |z|^2 = r^2,$$

$$z + \overline{z} = 2r\cos\theta,$$

$$z - \overline{z} = 2ri\sin\theta.$$

Thus, if $\omega^n = 1$ or -1, then for all $\omega_k = e^{i\theta_k}$,

$$(x - \omega_k)(x - \omega_{n-k}) = (x - \omega_k)(x - \overline{\omega_k}) = x^2 - 2x\cos\theta_k + 1,$$

$$(x + \omega_k)(x + \omega_{n-k}) = (x - \omega_k)(x + \overline{\omega_k}) = x^2 - 2xi\sin\theta_k - 1.$$

If we plug in $x = 1$ and take the product over all ω_k, we get

$$\prod_{k=1}^{n-1}(1 - \omega_k)(1 + \overline{\omega_k}) = (-2i)^{n-1}\prod_{k=1}^{n-1}\sin\theta_k.$$

as desired. □

Exercise 3.21. Derive a similar equation for cosines. **Hints**: 24

Theorem 3.22 (Triple Angle Trig Theorem)
Let A, B, C be angles such that

$$\sin A + \sin B + \sin C = \cos A + \cos B + \cos C = 0.$$

Then $3\cos(A + B + C) = \cos 3A + \cos 3B + \cos 3B$ and $3\sin(A + B + C) = \sin 3A + \sin 3B + \sin 3C$

Proof. This is one of the most coveted uses of complex numbers. We have that if we let

$$a = \cos A + i\sin A = e^{iA}$$

$$b = \cos B + i\sin B = e^{iB}$$

$$c = \cos C + i\sin C = e^{iC}$$

then $a + b + c = 0$. Noting the identity $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$, we get

$$a^3 + b^3 + c^3 = 3abc \implies e^{3iA} + e^{3iB} + e^{3iC} = 3e^{i(A+B+C)}$$

Breaking this into real and imaginary parts, we get the desired result. □

Example 3.23
Find $2\cos 72^\circ$.

Solution. Let $z = e^{\frac{2k\pi}{5}}$. This implies

$$z^5 = 1,$$

and $z \neq 1$, so

$$(z-1)(z^4+z^3+z^2+z+1) = 0,$$
$$z^4+z^3+z^2+z+1 = 0.$$

Note that $2\cos 72^\circ = z + \frac{1}{z}$. If we divide the equation above by z^2, we get

$$z^2+z+1+\frac{1}{z}+\frac{1}{z^2} = 0,$$
$$\left(z^2+\frac{1}{z^2}\right)+\left(z+\frac{1}{z}\right)+1 = 0,$$
$$\left(z+\frac{1}{z}\right)^2+\left(z+\frac{1}{z}\right)-1 = 0,$$

which implies

$$\left(z+\frac{1}{z}\right) = \boxed{\frac{-1+\sqrt{5}}{2}}.$$

Note that we find that the other root doesn't work from bounding $\cos 72^\circ$ (i.e. it is positive from 0° to 90°). $\square$

Example 3.24 (AIME II 2005/9)

For how many positive integers n less than or equal to 1000 is $(\sin t + i\cos t)^n = \sin nt + i\cos nt$ true for all real t?

Solution. We note that this looks a lot like De Moivre's Theorem - if only we could get it in that form! Well, we know

$$\sin x = \cos\left(\frac{\pi}{2}-x\right)$$

so we basically have

$$\cos n\left(\frac{\pi}{2}-t\right)+i\sin n\left(\frac{\pi}{2}-t\right) = (\cos\left(\frac{\pi}{2}-t\right)+i\sin\left(\frac{\pi}{2}-t\right)^n = (\sin t+i\cos t)^n = \sin nt + i\cos nt$$

Now, it really boils down to finding the solutions to $\cos x = \sin y$, right? But what are they? Fortunately, we can use our above observation again to get

$$\cos x = \cos\left(\frac{\pi}{2}-y\right)$$

Now, we note that $\cos a = \cos b$ if and only if $a-b$ or $a+b$ is a multiple of 2π. Thus, we get that

$$2\pi k = x+\frac{\pi}{2}-y$$

or

$$2\pi k = x+y-\frac{\pi}{2}$$

Thus, we have that either

$$2\pi k = n\left(\frac{\pi}{2}-t\right)+nt-\frac{\pi}{2} = \frac{(n+1)\pi}{2}$$
$$2\pi k = n\left(\frac{\pi}{2}-t\right)-nt-\frac{\pi}{2} = \frac{(n-1)\pi}{2}-2nt$$

The second can't hold for all t, so the first one is our only possibility. But that's not too hard to find - it's just all $n \equiv 1 \pmod 4$. This gives us $\boxed{250}$ solutions. $\square$

Example 3.25 (AIME I 2012/6)
The complex numbers z and w satisfy $z^{13} = w$, $w^{11} = z$, and the imaginary part of z is $\sin \frac{m\pi}{n}$, for relatively prime positive integers m and n with $m < n$. Find n.

Solution. Well, we should be easily able to do this. We have that $z = z^{11\cdot 13} = z^{143}$, so it is a 142 root of unity. Thus, it is in the form

$$\cos \frac{2\pi k}{142} + i \sin \frac{2\pi k}{142}$$

Thus, the answer is $\boxed{71}$ (don't forget to divide by 2!) □

Here's a few exercises to test you: none of them are from the AIME.

Exercise 3.26. What is the value of $\sin 20^\circ \sin 40^\circ \sin 80^\circ$? **Hints**: 15

Exercise 3.27 (Lagrange's Trigonometric Identity)**.** For all angles θ and positive integer n,

$$1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = \frac{1}{2} + \frac{\sin\left[(2n+1)\frac{\theta}{2}\right]}{2\sin\left(\frac{\theta}{2}\right)},$$

and derive a similar expression for sine. **Hints**: 28

Exercise 3.28 (Generalized ARML 2013)**.** Let $a = \cos\frac{2\pi}{7}, b = \cos\frac{4\pi}{7}$, and $c = \cos\frac{8\pi}{7}$. Then compute $ab + bc + ca$ and $a^3 + b^3 + c^3$. **Hints**: 76

Exercise 3.29 (PUMaC 2010/7)**.** The expression $\sin 2^\circ \sin 4^\circ \sin 6^\circ \ldots \sin 90^\circ$ is equal to $\frac{p\sqrt{5}}{2^{50}}$, where p is an integer. Find p. **Hints**: 71

Exercise 3.30. Let $\omega = e^{\frac{2\pi i}{101}}$. Evaluate the product

$$\prod_{0\le p<q\le 100} (\omega^p + \omega^q).$$

Hints: 41

Exercise 3.31. $ABCDEFG$ is a regular heptagon inscribed in a unit circle. Compute the value of the following expression:

$$AB^2 + AC^2 + AD^2 + AE^2 + AF^2 + AG^2.$$

Hints: 56 27

§4 Applications to Planar Geometry

§4.1 Direct Applications

Theorem 4.1 (Trigonometric Laws)
In triangle ABC with $a = BC, b = CA, c = AB$,

- Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- Law of Cosines: $a^2 = b^2 + c^2 - 2bc\cos A$
- Law of Tangents: $\frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)} = \frac{a-b}{a+b}$

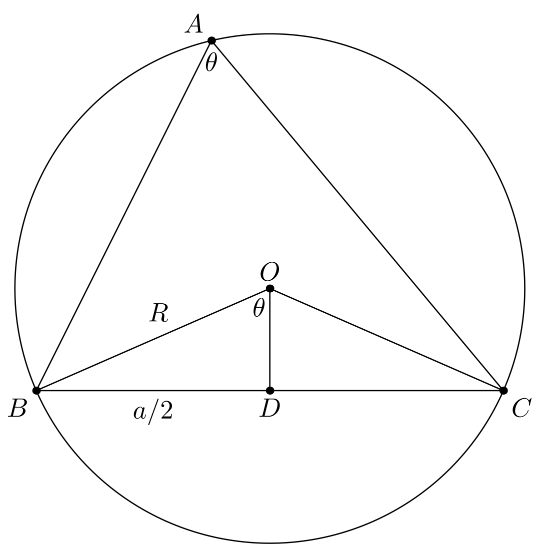
Theorem 4.2 (Extended Law of Sines)
Let ABC be a triangle with sides a, b, and c, and of circumradius R. Then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Proof. In the diagram below, point O is the circumcenter of $\triangle ABC$. Point D is on BC such that OD is perpendicular to BC. Since $\triangle ODB \cong \triangle ODC$, $BD = CD = \frac{a}{2}$ and $\angle BOD = \angle COD$. But $2\angle BAC = \angle BOC$ making $\angle BOD = \angle COD = \theta$. We can use simple trigonometry in right triangle $\triangle BOD$ to find that

$$\sin\theta = \frac{\frac{a}{2}}{R} \iff \frac{a}{\sin\theta} = 2R.$$

The same holds for b and c, thus establishing the identity.

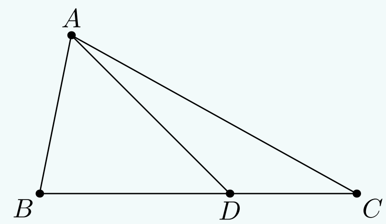


□

Law of Cosines has been listed before, so to avoid repetition I will not list it again.

Theorem 4.3 (Ratio Lemma)
Let D be a point on BC in triangle ABC. Then

$$\frac{DB}{DC} = \frac{AB}{AC} \cdot \frac{\sin \angle DAB}{\sin \angle DAC}.$$



Theorem 4.4 (Trig Ceva)
Let ABC be a triangle with points D, E, and F on sides BC, AC, and AB respectively of triangle ABC. Line segments AD, BE, and CF are concurrent if and only if

$$\frac{\sin \angle BAC \sin \angle ACF \sin \angle CBE}{\sin \angle DAC \sin \angle FCB \sin \angle EBA} = 1.$$

Sketch of Proof. From Regular Ceva's, start to apply Law of Sines everywhere fathomable. □

Theorem 4.5 (Quadratic Formula of Trigonometry)
Let

$$a \cos \theta + b \sin \theta = c.$$

Then

$$\cos \theta = \frac{ac \pm b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2},$$

$$\sin \theta = \frac{bc \pm \sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}$$

Proof. Just solve

$$a \cos \theta + b \sin \theta = c$$

and

$$\cos^2 \theta + \sin^2 \theta = 1$$

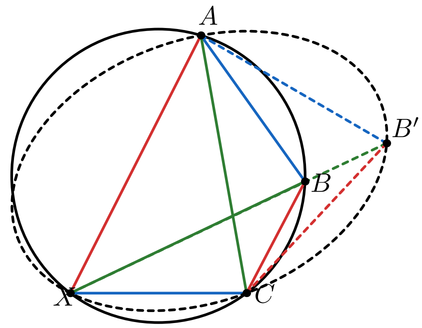
It's not fun, but it'll come with enough practice. □

The following theorem and proof is by AoPS User NJOY. Many thanks to him for the diagram and the proof!

Theorem 4.6 (Trigonometric Form of Ptolemy's Theorem)
Four points X, A, B, C of the Euclidean plane are concyclic if, and only if

$$XA \cdot \sin \angle BXC + XB \cdot \sin \angle CXA + XC \cdot \sin \angle AXB = 0.$$

Proof. Without Loss of Generality, we can assume that the ray $\overleftrightarrow{XB}$ lies between $\overleftrightarrow{XA}$ and $\overleftrightarrow{XC}$, as in the diagram below. Let B' be the point in which XB intersects the circle $\odot(XAC)$.



Then by Ptolemy's theorem,

$$XA \cdot CB' + XC \cdot AB' = XB' \cdot AC.$$

By the law of sines,

$$2R = \frac{AB'}{\sin \angle AXB} = \frac{B'C}{\sin \angle BXC} = \frac{AC}{\sin \angle CXA},$$

so that we get $XA \cdot \sin \angle BXC + XB' \cdot \sin \angle CXA + XC \cdot \sin \angle AXB = 0$. Therefore, $XB' = XB$ and hence, $B' = B$, as desired. □

Here is a nice problem using Trigonometric Form of Ptolemy's Theorem:

Example 4.7 (IMO SL 2012 G2)

Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.

Proof. Using Trigonometric Form of Ptolemy's Theorem, it remains to prove that

$$DH \cdot \sin \angle FDG + DG \cdot \sin \angle FDH = DF \cdot \sin \angle HDG.$$

Note that DH = DE and DG = CE. Now, simple angle chasing yields

$$\angle FDG = \angle DBC, \quad \angle HDF = \angle ADB, \quad \angle HDG = \angle DFC.$$

Our condition can then be rewritten as

$$DE \cdot \sin \angle DBC + CE \cdot \sin \angle ADB = DF \cdot \sin \angle DFC.$$

Now, by the law of sines,

$$DF \cdot \sin \angle DFC = DC \cdot \sin \angle FCD = DC \cdot \sin \angle BCD$$

$$CE \cdot \sin \angle ADB = CE \cdot \sin \angle ECB = EB \cdot \sin \angle EBC$$

and hence

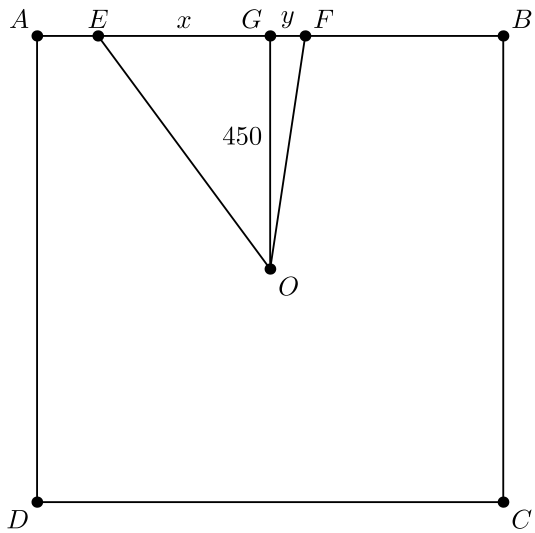
$$DE \cdot \sin \angle DBC + EB \cdot \sin \angle EBC = DC \cdot \sin \angle BCD,$$

as desired. □

Let's look at a few problems:

Example 4.8

Square $ABCD$ has center O, $AB = 900$, E and F are on AB with $AE < BF$ and E between A and F, $m\angle EOF = 45^\circ$, and $EF = 400$. Given that $BF = p + q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of any prime, find $p + q + r$.



Solution. We let G be the midpoint of AB. We get

$$\tan\angle EOG = \frac{EG}{OG} = \frac{EG}{450}$$

$$\tan\angle FOG = \frac{FG}{OG} = \frac{FG}{450}$$

Thus, we can use the tangent addition formula to get

$$\tan\angle EOF = \frac{\frac{FG}{450} + \frac{EG}{450}}{1 - \frac{FG\cdot EG}{450^2}}$$

But wait! $\tan\angle EOF = \tan 45^\circ = 1$! So we have that

$$450^2 - FG \cdot EG = 450 \cdot EF = 450 \cdot 400$$

Thus, we get

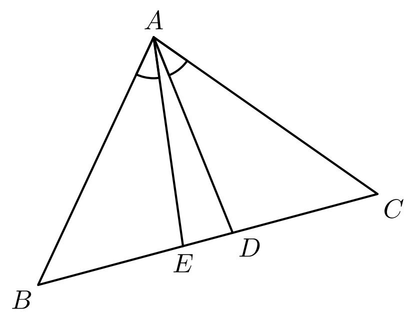
$$FG \cdot EG = 150^2$$

$$FG + EG = 400$$

We can get a quadratic by substitution - we have $FG^2 - 400FG + 150^2 = 0$, so $FG = 200 \pm 50\sqrt{7}$ (the $\pm$ is there to choose between FG and EG). However, $FG < EG$, so $FG = 200 - 50\sqrt{7}$, so $BF = 250 + 50\sqrt{7}$. The answer is thus $\boxed{307}$. □

Example 4.9 (AIME II 2005/14)

In triangle $ABC, AB = 13, BC = 15$, and $CA = 14$. Point D is on $\overline{BC}$ with $CD = 6$. Point E is on $\overline{BC}$ such that $\angle BAE \cong \angle CAD$. Given that $BE = \frac{p}{q}$ where p and q are relatively prime positive integers, find q.



Solution. There are very smart solutions using the fact that AE and AD are isogonal. However, that isn't really enough for us. We shall trig bash. Using the ratio lemma, we have

$$\frac{BD}{DC} = \frac{AB}{AC} \cdot \frac{\sin \angle BAD}{\sin \angle CAD}$$

What's the inspiration for this? Well, the ratio lemma is always a handy tool. And we have that $\angle BAE = \angle CAD$ and $\angle BAD = \angle CAE$, which is even nicer. So that's why we try this. Similarly, we get

$$\frac{BE}{EC} = \frac{AB}{AC} \cdot \frac{\sin \angle BAE}{\sin \angle CAE}$$

so multiplying get's rid of our worries

$$\frac{BE \cdot BD}{CD \cdot EC} = \frac{AB^2}{AC^2}$$

This implies

$$\frac{CE}{BE} = \frac{3 \cdot 14^2}{2 \cdot 13^2} = \frac{294}{169}$$

Now, we look for BE. Noting $CE + BE = BC$, we get

$$\frac{15}{BE} - 1 = \frac{BC}{BE} - 1 = \frac{CE}{BE} = \frac{294}{169}$$

Thus, we solve for BE to get

$$BE = \frac{2535}{463}$$

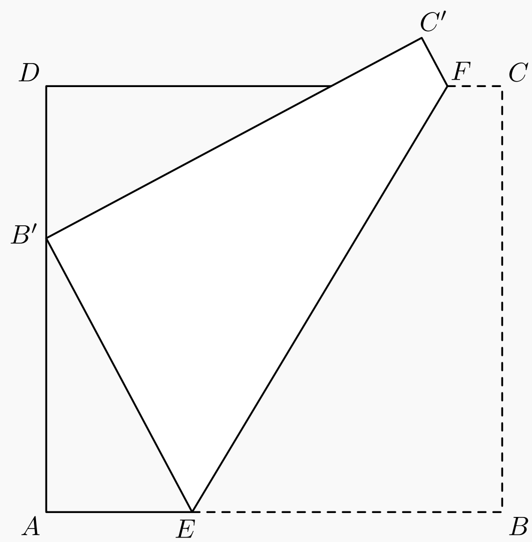
which gives the answer as $\boxed{463}$. □

Try the next few problems:

Exercise 4.10 (AIME 1987/15)**.** Squares S_1 and S_2 are inscribed in right triangle ABC, as shown in the figures below. Find $AC + CB$ if area $(S_1) = 441$ and area $(S_2) = 440$. **Hints**: 51 77

Exercise 4.11 (AIME 1985/9)**.** In a circle, parallel chords of lengths 2, 3, and 4 determine central angles of α, β, and $\alpha + \beta$ radians, respectively, where $\alpha + \beta < \pi$. If $\cos \alpha$, which is a positive rational number, is expressed as a fraction in lowest terms, what is the sum of its numerator and denominator? **Hints**: 98

Exercise 4.12 (AIME II 2004/7)**.** $ABCD$ is a rectangular sheet of paper that has been folded so that corner B is matched with point B' on edge AD. The crease is EF, where E is on AB and F is on CD. The dimensions $AE = 8, BE = 17$, and $CF = 3$ are given. The perimeter of rectangle $ABCD$ is m/n, where m and n are relatively prime positive integers. Find $m + n$. **Hints**: 62



§4.2 Indirect Applications

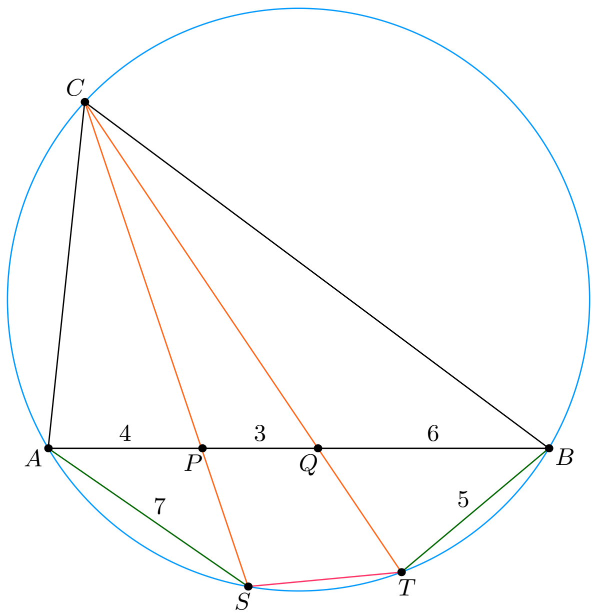
What is this section for? Well, sometimes when working with problems, trigonometry sometimes doesn't solve the whole problem - you have to combine it with techniques such as coordinates and/or synthetic observations, that will finish the problem. Let's see some examples:

Example 4.13 (AIME II 2016/10)

Triangle ABC is inscribed in circle ω. Points P and Q are on side $\overline{AB}$ with $AP < AQ$. Rays CP and CQ meet ω again at S and T (other than C), respectively. If $AP = 4, PQ = 3, QB = 6, BT = 5$, and $AS = 7$, then $ST = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. I'll provide a proof for those who are an expert at projective geometry - take a pencil through C and the cross ratio is preserved.

For those of you who didn't understand that, let's use the Ratio Lemma:



Well, let's try to find almost everything here. Well, we note that we have a ton of application of similar triangles (by, say AA similarity):

- $\triangle ACQ \sim \triangle TBQ$
- $\triangle PCB \sim \triangle PAS$

There are more, but those are enough. In fact, the first one itself suffices. We get from Ratio Lemma that

$$\frac{AC}{CQ} = \frac{5}{6}$$

so we get that using Ratio lemma on $\triangle ACQ$, we have

$$\frac{4}{3} = \frac{AP}{PQ} = \frac{AC}{CQ} \cdot \frac{\sin \angle ACP}{\sin \angle PCQ} = \frac{5}{6} \cdot \frac{\sin \angle ACP}{\sin \angle PCQ}$$

This implies that

$$\frac{\sin \angle ACP}{\sin \angle PCQ} = \frac{24}{15}$$

Well, I could have kept on going, but it's not of substance anymore. We can just use the Extended Law of Sines to get

$$\frac{AS}{\sin \angle ACP} = 2R = \frac{ST}{\sin \angle PCQ}$$

which implies

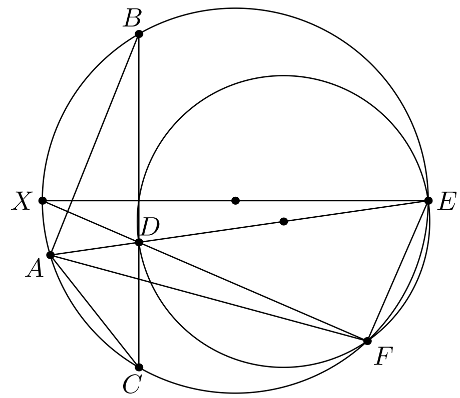
$$ST = AC \cdot \frac{\sin \angle ACP}{\sin \angle PCQ} = \frac{35}{8}$$

so the answer is $35 + 8 = \boxed{43}$. □

Example 4.14 (AIME II 2012/15)

Triangle ABC is inscribed in circle ω with $AB = 5$, $BC = 7$, and $AC = 3$. The bisector of angle A meets side $\overline{BC}$ at D and circle ω at a second point E. Let γ be the circle with diameter $\overline{DE}$. Circles ω and γ meet at E and a second point F. Then $AF^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. For those of you who know how inversions and harmonics work, here's a two-second solution: Use a force overlaid $\sqrt{bc}$ inversion to get $M \longleftrightarrow F$ (M is the midpoint of BC) implying AF is the symmedian so $ABFC$ is a harmonic bundle. Law of Cosines and/or Ptolemy's will finish it off.



> **Remark 4.15.** While this solution is quick, it is *overkill.* Try not to use the sledgehammer when only a hammer is needed. That being said, don't use a hammer when nothing needs to be nailed. However, if something *does* need to be nailed, use a hammer.

Now, if you are a normal competitor, and have no idea what that means, don't worry. This is trig, so I'll talk about trig here. First, what's $\triangle ABC$? That seems like horrible numbers - unless we have nice numbers. We try to use the law of cosines:

$$\cos\angle A = \frac{5^2+3^2-7^2}{2\cdot 5\cdot 3} = \frac{-15}{2\cdot 15} = -\frac{1}{2}$$
$$\cos\angle B = \frac{5^2+7^2-3^2}{2\cdot 5\cdot 7} = \frac{65}{2\cdot 35} = \frac{13}{14}$$
$$\cos\angle C = \frac{3^2+7^2-5^2}{2\cdot 3\cdot 7} = \frac{34}{2\cdot 21} = \frac{17}{21}$$

Well, only the first one looks nice. That's $\angle A = 120^\circ$, and seems like the only nice characterization here. Now, this also makes sense as the problem is A-indexed. Now, let's see what else we have. We note that as E is on the angle bisector, we have

$$\angle EAB = \angle EAC = 60^\circ$$

Why did we choose E? Well, it's on the circumcircle, so it's nice right away, and another important property is that E is the midpoint of arc BC. That's because we have that

$$\angle CBE = \angle CAE = \angle BAE = \angle BCE$$

which means $CE = BE$. But wait! We know that's 60°, so we have an equilateral triangle. What other information can we extract? Well, we know that we have to get F into the picture somehow. But how do we insert F? Well, what do we know? Looking at our characterizations, we have that $\angle DFE = 90^\circ$. What else do we know? $\angle BFC = 60^\circ$, but that doesn't give us any important things. Remember, if we can find most of these lengths, from the diagram, it appears we should apply Ptolemy to $ABFC$. So we need to find BF and FC. How do we do that? Well, time to find almost every single length possible. First, let's find CD, BD, AD, as we have good control of those. We use the angle bisector theorem to get

$$BD = \frac{AB}{AB+AC}\cdot BC = \frac{35}{8}$$
$$CD = \frac{AC}{AB+AC}\cdot BC = \frac{21}{8}$$

Now, AD - that's Stewart's theorem, right? We get

$$AD = \sqrt{\frac{1}{BC}\left(AC^2 \cdot BM + AB^2 \cdot CM - CM \cdot BM \cdot AB\right)} = \sqrt{\frac{225}{64}} = \frac{15}{8}$$

So everything is a rational number! How nice. Let's write down every single Law of Cosines equation we can get:

$$AE^2 = AF^2 + EF^2 - 2 \cdot AF \cdot EF \cos\angle AFE$$

$$AF^2 = AE^2 + EF^2 - 2 \cdot AE \cdot EF \cos\angle AEF$$

What can we do? Canel stuff out! We get

$$2 \cdot EF^2 = 2 \cdot EF \cdot (AF \cos\angle AFE + AE \cos\angle AEF)$$

Hmmm...what do we know about those angles? well, we know that E, F is on the circle, so $\angle AFE = \angle ACE$. We're doing this to get rid of F, so let's see if we can get $\angle AEF$. Well, we know that D, E, A are collinear, so $\angle AEF = \angle DEF$. But that's perfect! We know that as DEF is a right triangle (right angle at F) so we have that

$$\cos\angle AEF = \cos\angle DEF = \frac{EF}{DE} = \frac{8EF}{49}$$

$$\cos\angle AFE = \cos\angle ACE = \frac{AC^2 + CE^2 - AE^2}{2 \cdot AC \cdot CE} = \frac{AC^2 + BC^2 - AE^2}{2 \cdot AC \cdot BC} = -\frac{1}{7}$$

So we can find EF! We have

$$EF = -\frac{AF}{7} + \frac{8 \cdot EF \cdot AE}{64} = -\frac{AF}{7} + \frac{64EF}{49}$$

which implies that

$$15EF = 7AF$$

Well, we were so close! But we also have that

$$AE^2 = AF^2 + EF^2 - 2 \cdot AF \cdot EF \cos\angle AFE$$

Let's use this equation one last time! Substituting, we get

$$64 = 8^2 = AE^2 = AF^2 + EF^2 - 2 \cdot AF \cdot EF \cos\angle AFE = AF^2 + \frac{49}{225}AF^2 + \frac{2}{15}AF^2 = \frac{304}{225}AF^2$$

We get

$$AF^2 = \frac{900}{19}$$

which gives us an answer of $\boxed{919}$. □

So what exactly did we do? Made a few trivial observations, and then said hello to our good friend Law of Cosines. Sometimes, it will be Law of Sines, but it's not always too much of a variant. There are only 2 main laws, after all! In addition, this example was pretty involved - make sure you understand this.

Exercise 4.16 (AIME II 2003/14)**.** Let $A = (0, 0)$ and $B = (b, 2)$ be points on the coordinate plane. Let $ABCDEF$ be a convex equilateral hexagon such that $\angle FAB = 120^\circ$, $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and the y-coordinates of its vertices are distinct elements of the set $\{0, 2, 4, 6, 8, 10\}$. The area of the hexagon can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$. **Hints**: 91 68 43

Exercise 4.17 (AIME I 2018/15)**.** David found four sticks of different lengths that can be used to form three non-congruent convex cyclic quadrilaterals, A, B, C, which can each be inscribed in a circle with radius 1. Let φ_A denote the measure of the acute angle made by the diagonals of quadrilateral A, and define φ_B and φ_C similarly. Suppose that $\sin\varphi_A = \frac{2}{3}$, $\sin\varphi_B = \frac{3}{5}$, and $\sin\varphi_C = \frac{6}{7}$. All three quadrilaterals have the same area K, which can be written in the form $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$. **Hints**: 48 67 6

Exercise 4.18 (AIME I 2005/15)**.** Triangle ABC has $BC = 20$. The incircle of the triangle evenly trisects the median AD. If the area of the triangle is $m\sqrt{n}$ where m and n are integers and n is not divisible by the square of a prime, find $m + n$. **Hints**: 11 84 85

By the way, I'm sorry for all the hard problem here!

§4.3 Trigonometric Functions at Special Values

There are a few special angles for which you should know the values of the trigonometric functions, without having to resort to a table or a calculator. These are summarized in the following table.

θ	$\cos(\theta)$	$\sin(\theta)$
0	1	0
$\dfrac{\pi}{12}$	$\dfrac{\sqrt{6}-\sqrt{2}}{4}$	$\dfrac{\sqrt{6}+\sqrt{2}}{4}$
$\dfrac{\pi}{10}$	$\dfrac{\sqrt{5}-1}{4}$	$\sqrt{\dfrac{5+\sqrt{5}}{8}}$
$\dfrac{\pi}{8}$	$\dfrac{\sqrt{2-\sqrt{2}}}{2}$	$\dfrac{\sqrt{2+\sqrt{2}}}{2}$
$\dfrac{\pi}{5}$	$\sqrt{\dfrac{5-\sqrt{5}}{8}}$	$\dfrac{\sqrt{5}+1}{4}$
$\dfrac{\pi}{6}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{2}$
$\dfrac{\pi}{4}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$
$\dfrac{\pi}{3}$	$\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$
$\dfrac{\pi}{2}$	0	1

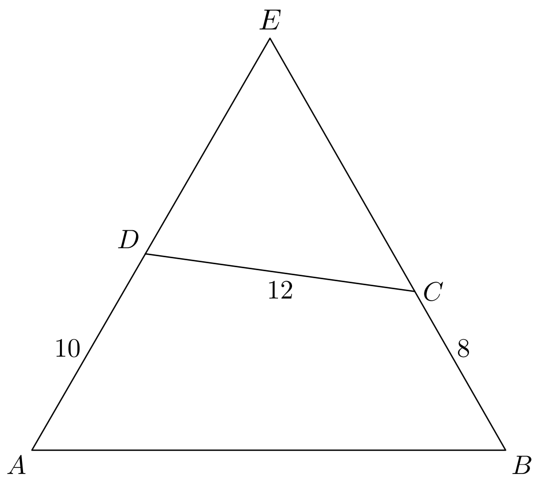
You should also be able to use reference angles along with these values to compute the values of the trigonometric functions at related angles in the second, third and fourth quadrants. For example, the point associated with $t = \frac{5\pi}{6}$ is directly across the unit circle from the point associated with $\frac{\pi}{6}$. (In this case we say that we are using

$\frac{\pi}{6}$ as a reference angle.) Thus the coordinates of the point associated with $\frac{5\pi}{6}$ has the same y value and the opposite x value of the point associated with $\frac{\pi}{6}$. Thus $\cos(\frac{5\pi}{6}) = -\cos(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}$, and $\sin(\frac{5\pi}{6}) = \sin(\frac{\pi}{6}) = \frac{1}{2}$.

This especially useful for geometry problems in which the angle is given, and the angles are nice. Memorizing special properties of certain triangles is extremely useful. One of the first things you should try when parts of a triangle are given is to look for special angles.

Example 4.19 (AIME I 2005/7)

In quadrilateral $ABCD$, $BC = 8$, $CD = 12$, $AD = 10$, and $m\angle A = m\angle B = 60^\circ$. Given that $AB = p + \sqrt{q}$, where p and q are positive integers, find $p + q$.



Solution. Extend the lines to meet at E. This is a fairly standard trick, and has many useful results. In this case, $\triangle ABE$ is isosceles, so $AE = BE$. Now, if you know what we're planning on doing, it's time for the Law of Cosines. Assuming $AE = BE = x$, we get that $DE = x - 10$ and $CE = x - 8$, so we get (as $\angle AEB = 60^\circ$)

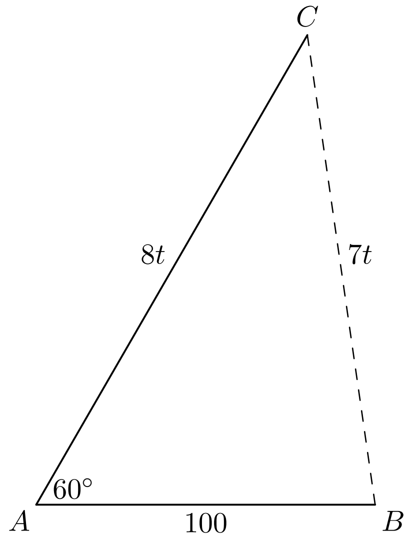
$$144 = 12^2 = CD^2 = DE^2 + CE^2 - 2 \cdot DE \cdot CE \cos \angle DEC = (x-10)^2 + (x-8)^2 - 2 \cdot (x-10)(x-8) \cdot \frac{1}{2} = x^2 - 18x + 84$$

We get two roots - $x = 9 \pm \sqrt{141}$. Which one do we take?

We note that $9 - \sqrt{141}$ is negative, so it's $9 + \sqrt{141}$. The answer is $9 + 141 = \boxed{150}$. □

Example 4.20 (AIME 1989/6)

Two skaters, Allie and Billie, are at points A and B, respectively, on a flat, frozen lake. The distance between A and B is 100 meters. Allie leaves A and skates at a speed of 8 meters per second on a straight line that makes a 60° angle with AB. At the same time Allie leaves A, Billie leaves B at a speed of 7 meters per second and follows the straight path that produces the earliest possible meeting of the two skaters, given their speeds. How many meters does Allie skate before meeting Billie?



Solution. Well, this sort of is a $d = rt$ problem. Of course we're throwing in the Law of Cosines! We have that as seen in the above diagram

$$49t^2 = (7t)^2 = (8t)^2 + 100^2 - 2 \cdot 8t \cdot 100^2 \cos 60^\circ = 64t^2 - 800t + 10000$$

This rearranges to

$$15t^2 - 800t + 10000 = 0$$

which by the quadratic formula gives

$$t = \frac{100}{3}, 20$$

Wait - there are two answers. How do we choose? Well, we must have the first intersection - so the answer is $8 \cdot 2 = \boxed{160}$. An alternative (that would work on a test) would be to use the fact AIME has integral answers. □

Theorem 4.21 (Blanchet's Theorem)

Let AD, BE, and CF be concurrent cevians in $\triangle ABC$. If $AD \perp BC$, show that ray AD bisects $\angle EDF$.

Proof. Note that

$$\begin{aligned}
\frac{\tan\angle ADE}{\tan\angle ADF} &= \frac{\sin\angle ADE}{\sin\angle ADF} \cdot \frac{\cos\angle ADF}{\cos\angle ADE} \\
&= \frac{\sin\angle ADE}{\sin\angle ADF} \cdot \frac{\sin\angle FDB}{\sin\angle EDC} \\
&= \frac{\frac{AE}{AD}\sin\angle AED}{\frac{AF}{AD}\sin\angle AFD} \cdot \frac{\sin FDB}{\sin EDC} \\
&= \frac{AE}{AF} \cdot \frac{\sin\angle CED}{\sin\angle BFD} \cdot \frac{\sin\angle FDB}{\sin\angle EDC} \\
&= \frac{AE}{AF} \cdot \frac{\sin\angle CED}{\sin\angle EDC} \cdot \frac{\sin\angle FDB}{\sin\angle BFD} \\
&= \frac{AE}{AF} \cdot \frac{CD}{CE} \cdot \frac{FB}{BD} \\
&= \frac{AE}{CE} \cdot \frac{CD}{BD} \cdot \frac{FB}{AF} \\
&= 1
\end{aligned}$$

by Ceva's theorem, so $\angle ADE = \angle ADF$. □

Exercise 4.22 (AIME II 2004/1)**.** A chord of a circle is perpendicular to a radius at the midpoint of the radius. The ratio of the area of the larger of the two regions into which the chord divides the circle to the smaller can be expressed in the form $\frac{a\pi+b\sqrt{c}}{d\pi-e\sqrt{f}}$, where a, b, c, d, e, and f are positive integers, a and e are relatively prime, and neither c nor f is divisible by the square of any prime. Find the remainder when the product $abcdef$ is divided by 1000. **Hints**: 9

Exercise 4.23 (AIME 1990/12)**.** A regular 12-gon is inscribed in a circle of radius 12. The sum of the lengths of all sides and diagonals of the 12-gon can be written in the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, and d are positive integers. Find $a + b + c + d$. **Hints**: 89

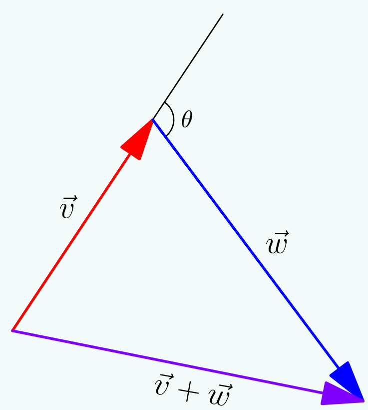
Exercise 4.24 (1995 AIME/14)**.** In a circle of radius 42, two chords of length 78 intersect at a point whose distance from the center is 18. The two chords divide the interior of the circle into four regions. Two of these regions are bordered by segments of unequal lengths, and the area of either of them can be expressed uniquely in the form $m\pi - n\sqrt{d}$, where m, n, and d are positive integers and d is not divisible by the square of any prime number. Find $m + n + d$. **Hints**: 81 8 36

§4.4 Vector Geometry

Definition 4.25 (Vector) — A **vector** is a directed line segment. It can also be considered a quantity with magnitude and direction. Every vector $\overrightarrow{UV}$ has a starting point $U\langle x_1, y_1\rangle$ and an endpoint $V\langle x_2, y_2\rangle$.

Theorem 4.26 (Addition of Vectors)

For vectors $\vec{v}$ and $\vec{w}$, with angle θ formed by them, $\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\|\cos\theta$.



Theorem 4.27 (Multiplying Vectors by Constant)

For some constant $c > 0$, $c\vec{v}$ increases the magnitude of $\vec{v}$ by c times in the same direction as $\vec{v}$. If $c < 0$, $c\vec{v}$ increases the magnitude of $\vec{v}$ by c times in the opposite direction as $\vec{v}$. If $c = 0$, the magnitude becomes 0 and there is no direction.

Theorem 4.28 (Vector Identities)
For any vectors $\vec{x}$, $\vec{y}$, $\vec{z}$, and real numbers a, b,

1. **Commutative Property**: $\vec{x}+\vec{y}=\vec{y}+\vec{x}$
2. **Associative Property**: $(\vec{x}+\vec{y})+\vec{z}=\vec{x}+(\vec{y}+\vec{z})$
3. **Additive Identity**: There exists the zero vector $\vec{0}$ such that $\vec{x}+\vec{0}=\vec{x}$
4. **Additive Inverse**: For each $\vec{x}$, there is a vector $\vec{y}$ such that $\vec{x}+\vec{y}=\vec{0}$
5. **Unit Scalar Identity**: $1\vec{x}=\vec{x}$
6. **Associative in Scalar**: $(ab)\vec{x}=a(b\vec{x})$
7. **Distributive Property of Vectors**: $a(\vec{x}+\vec{y})=a\vec{x}+a\vec{y}$
8. **Distributive Property of Scalars**: $(a+b)\vec{x}=a\vec{x}+b\vec{x}$

Definition 4.29 (Dot Product) — Consider two vectors $\mathbf{a}=\langle a_1, a_2, \ldots, a_n\rangle$ and $\mathbf{b}=\langle b_1, b_2, \ldots, b_n\rangle$ in $\mathbb{R}^n$. The **dot product** is equal to the length of the projection (i.e. the distance from the origin to the foot of the head of $\mathbf{a}$ to $\mathbf{b}$) of $\mathbf{a}$ onto $\mathbf{b}$ times the length of $\mathbf{b}$.

Theorem 4.30 (Magnitude of Dot Product)
Consider two vectors $\mathbf{a}=\langle a_1, a_2, \ldots, a_n\rangle$ and $\mathbf{b}=\langle b_1, b_2, \ldots, b_n\rangle$ in $\mathbb{R}^n$. The dot product is then

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}=|\mathbf{a}||\mathbf{b}|\cos\theta=a_1b_1+a_2b_2+\cdots+a_nb_n,$$

where θ is the angle formed by the two vectors.

Definition 4.31 (Cross Product) — The **cross product** between two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ is defined as the vector whose length is equal to the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ and whose direction is in accordance with the right-hand rule.

Theorem 4.32 (Magnitude of Cross Product)
The magnitude of the cross product is

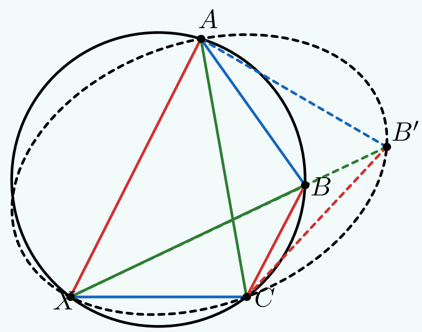
$$|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta,$$

where θ is the angle formed by the two vectors.

Vectors seem useless, but just take a look at this theorem by AoPS User A-Student. Thanks for helping out!

Theorem 4.33 (Vectorial Form of Ptolemy's Theorem)
With points X, A, B, C, B' as defined in the diagram below,

$$\frac{\vec{XA} \times \vec{XC}}{|\vec{XA}||\vec{XC}|} \cdot \frac{\vec{BX} \cdot \vec{BA}}{|\vec{BX}||\vec{BA}|} + \frac{\vec{XA} \cdot \vec{XC}}{|\vec{XA}||\vec{XC}|} \cdot \frac{\vec{BX} \times \vec{BA}}{|\vec{BX}||\vec{BA}|} = \frac{\vec{XB} \times \vec{XC}}{|\vec{XB}||\vec{XC}|}.$$



Proof. Four points X, A, B, C of the Euclidean plane are concyclic if and only if

$$XA \cdot \sin\angle BXC + XB \cdot \sin\angle CXA + XC \cdot \sin\angle AXB = 0.$$

We consider the reverse of this, we get:

If four points X, A, B, C of the Euclidean plane are concyclic, then

$$XA \cdot \sin\angle BXC + XB \cdot \sin\angle CXA + XC \cdot \sin\angle AXB = 0.$$

Let $\vec{XA}, \vec{XB}, \vec{XC}$ be position vectors of the points A, B & C taking X as the origin.
Now, as per property of concyclic quadrilaterals,

$$\angle AXC + \angle ABC = \pi$$

$$\implies \angle AXC + \angle ABX + \angle BXC = \pi$$

$$\implies \angle AXC + \angle ABX = \pi - \angle BXC.$$

$$\implies \sin(\angle AXC + \angle ABX) = \sin(\pi - \angle BXC).$$

$$\implies \sin(\angle AXC + \angle ABX) = \sin(\angle BXC).$$

Expanding the angles as per sin(A+B) rule,

$$\sin(\angle AXC)\cos(\angle ABX) + \cos(\angle AXC)\sin(\angle ABX) = \sin(\angle BXC).$$

$$\implies \frac{\vec{XA} \times \vec{XC}}{|\vec{XA}||\vec{XC}|} \cdot \frac{\vec{BX} \cdot \vec{BA}}{|\vec{BX}||\vec{BA}|} + \frac{\vec{XA} \cdot \vec{XC}}{|\vec{XA}||\vec{XC}|} \cdot \frac{\vec{BX} \times \vec{BA}}{|\vec{BX}||\vec{BA}|} = \frac{\vec{XB} \times \vec{XC}}{|\vec{XB}||\vec{XC}|}.$$

□

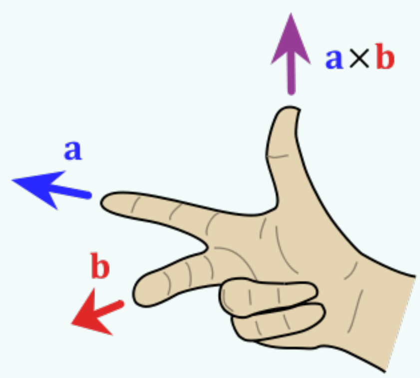
Exercise 4.34. Show that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$. **Hints**: 31

Exercise 4.35. Show that $|\mathbf{a}|^2|\mathbf{b}|^2 = |\mathbf{a} \cdot \mathbf{b}|^2 + |\mathbf{a} \times \mathbf{b}|^2$. **Hints**: 10 75

Exercise 4.36. A ship is travelling at a speed of 4 m/s to the north. A boy on the ship travels to the east at 3 m/s with respect to the ship. What speed does he travel at with respect to the sea (which is not moving)? **Hints**: 83

Theorem 4.37 (Right Hand Rule)

The **right hand rule** is used to determine the direction of the cross product. One can see this by holding one's hands outward and together, palms up, with the fingers curled, and the thumb out-stretched. If the curl of the fingers represents a movement from the first or x-axis to the second or y-axis, then the third or z-axis can point along either thumb.



Theorem 4.38 (Triple Scalar Product)

The triple scalar product of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is defined as $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Geometrically, the triple scalar product gives the signed volume of the parallelepiped determined by $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. It follows that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}.$$

Theorem 4.39 (Triple Vector Product)

The vector triple product of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is defined as the cross product of one vector, so that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$, which can be remembered by the mnemonic "BAC-CAB".

While the above theorems are extremely useful, the only crucial piece (for the AIME) is the following:

Theorem 4.40 (AIME Vectors)
Let θ be the angle between $\vec{u}$ and $\vec{v}$. Then

$$\vec{u} \cdot \vec{v} = uv\cos\theta,$$

and

$$|\vec{u} \times \vec{v}| = uv\sin\theta.$$

Theorem 4.41 (Properties of Vectors)
Some geometric properties of vectors:

1. If and only if the dot product of two vectors is zero, then those vectors are orthogonal or perpendicular. (The zero vector is orthogonal to every vector.)
2. If and only if the cross product of two vectors is zero (the zero vector), then those vectors are parallel. They can point in the same direction or in opposite directions.
3. The cross product of $\vec{u}$ and $\vec{v}$ is always orthogonal to $\vec{u}$ and $\vec{v}$. As long as $\vec{u}$ and $\vec{v}$ are not parallel, there exists one unique axis perpendicular to both which $\vec{u} \times \vec{v}$ will lie on.

Example 4.42 (AMC 10 A 2012/21)
Let points $A = (0,0,0)$, $B = (1,0,0)$, $C = (0,2,0)$, and $D = (0,0,3)$. Points E, F, G, and H are midpoints of line segments $\overline{BD}$, $\overline{AB}$, $\overline{AC}$, and $\overline{DC}$ respectively. What is the area of $EFGH$?

(A) $\sqrt{2}$ **(B)** $\frac{2\sqrt{5}}{3}$ **(C)** $\frac{3\sqrt{5}}{4}$ **(D)** $\sqrt{3}$ **(E)** $\frac{2\sqrt{7}}{3}$

Solution. Computing the points of $EFGH$ gives $E(0.5,0,1.5), F(0.5,0,0), G(0,1,0), H(0,1,1.5)$. The vector EF is $(0,0,-1.5)$, while the vector HG is also $(0,0,-1.5)$, meaning the two sides EF and GH are parallel. Similarly, the vector FG is $(-0.5,1,0)$, while the vector EH is also $(-0.5,1,0)$. Again, these are equal in both magnitude and direction, so FG and EH are parallel. Thus, figure $EFGH$ is a parallelogram.

Computation of vectors EF and HG is sufficient evidence that the figure is a parallelogram, since the vectors are not only point in the same direction, but are of the same magnitude, but the other vector FG is needed to find the angle between the sides.

Taking the dot product of vector EF and vector FG gives $0 \cdot -0.5 + 0 \cdot 1 + -1.5 \cdot 0 = 0$, which means the two vectors are perpendicular. (Alternately, as above, note that vector EF goes directly down on the z-axis, while vector FG has no z-component and lie completely in the xy plane.) Thus, the figure is a parallelogram with a right angle, which makes it a rectangle. With the distance formula in three dimensions, we find that $EF = \frac{3}{2}$ and $FG = \frac{\sqrt{5}}{2}$, giving an area of $\frac{3}{2} \cdot \frac{\sqrt{5}}{2} = \boxed{\textbf{(C)}\ \frac{3\sqrt{5}}{4}}$. □

§4.5 Parameterization

Parameterization is extremely useful for changing to only one variable, especially for conic sections.

Theorem 4.43 (Parameterizations of Conic Sections)
The following is the parametric equations for conic sections:

1. **circle:** $x = \sin\theta, y = \cos\theta$
2. **ellipse:** $x = a\sin\theta, y = b\cos\theta$
3. **hyperbola:** $x = a\sec\theta, y = b\tan\theta$
4. **parabola:** $x = 2pt^2, y = 2pt$

Note that the parameter for the parabola is t, because using an angle is mostly useless for parabolas.

Parameterization is also heavily influenced by complex numbers.

Theorem 4.44 (Polar Form of Conic Sections)
Let a focal point of a conic section lie at the origin. Then its polar form is

$$r = \frac{l}{1 - e\cos\theta},$$

where e is the eccentricity, and l is a constant. If:

1. $e = \mathbf{0}$: the equation is a circle
2. $\mathbf{0} < e < \mathbf{1}$: the equation is an ellipse
3. $e = \mathbf{1}$: the equation is a parabola
4. $e > \mathbf{1}$: the equation is a hyperbola

The parabola can also be determined by its trajectory:

Theorem 4.45 (Trajectory of a Parabola)
The trajectory of a parabola is given by

$$x \cdot \tan\theta \left(1 - \frac{x}{R}\right),$$

for constants θ and R.

Example 4.46 (AIME 1983/4)
A machine-shop cutting tool has the shape of a notched circle, as shown. The radius of the circle is $\sqrt{50}$ cm, the length of AB is 6 cm and that of BC is 2 cm. The angle ABC is a right angle. Find the square of the distance (in centimeters) from B to the center of the circle.

Solution. Draw segment OB with length x, and draw radius OQ such that OQ bisects chord AC at point M. This also means that OQ is perpendicular to AC. By the Pythagorean Theorem, we get that $AC = \sqrt{(BC)^2 + (AB)^2} = 2\sqrt{10}$, and therefore $AM = \sqrt{10}$. Also by the Pythagorean theorem, we can find that $OM = \sqrt{50 - 10} = 2\sqrt{10}$.

Next, find $\angle BAC = \arctan\left(\frac{2}{6}\right)$ and $\angle OAM = \arctan\left(\frac{2\sqrt{10}}{\sqrt{10}}\right)$. Since $\angle OAB = \angle OAM - \angle BAC$, we get

$$\angle OAB = \arctan 2 - \arctan\frac{1}{3}$$

$$\tan(\angle OAB) = \tan(\arctan 2 - \arctan\frac{1}{3})$$

By the subtraction formula for tan, we get

$$\tan(\angle OAB) = \frac{2-\frac{1}{3}}{1+2\cdot\frac{1}{3}}$$

$$\tan(\angle OAB) = 1$$

$$\cos(\angle OAB) = \frac{1}{\sqrt{2}}$$

Finally, by the Law of Cosines on $\triangle OAB$, we get

$$x^2 = 50 + 36 - 2(6)\sqrt{50}\frac{1}{\sqrt{2}}$$

$$x^2 = \boxed{026}.$$

□

Example 4.47

In acute angled triangle ABC, from a point D is on segment BC, draw perpendiculars DP and DQ to AB and AC, respectively. Show that PQ is minimized when D is the foot of the altitude from A to BC.

Solution. Let E be the foot of the altitude from A to BC and let R and S be the feet of the altitudes of AB and AC from E, respectively. I claim $APEDQ$ is cyclic. I will now prove this claim. Note that $\angle APD = \angle AED = \angle AQD = 90°$. Thus, not only does the circle $(APEDQ)$ exist, but AD is the diameter. I will now provide some motivation for the next result.

We note that at AE, AD is minimized. Because we are trying to prove PQ is minimized when $D = E$, it seems that as D approaches E (i.e. AD becomes minimized), PQ is minimized. This of course implies some relation between AD and PQ, motivating us to find the relation between these two.

Now that motivation is resolved, let us see how we can relate the two lengths. Let $\angle PAD = \alpha$ and $\angle DAQ = \beta$. Then using trigonometric identities, we have

$$AP = AD\cos\alpha, PD = AD\sin\alpha,$$

$$AQ = AD\cos\beta, QD = AD\sin\beta.$$

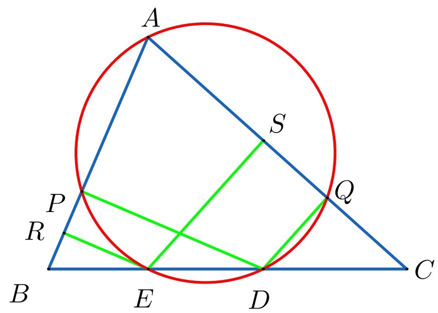
Using Ptolemy's Theorem on quadrilateral $APDQ$, we get

$$AD\cdot PQ = AP\cdot QD + AQ\cdot PD = AD\cos\alpha\cdot AD\sin\beta + AD\cos\beta\cdot AD\sin\alpha = AD^2\sin(\alpha+\beta),$$

and since $\alpha+\beta = \angle A$, we have

$$PQ = AD\sin\angle A.$$

Thus, as AD decreases, so does PQ, implying that when $D = E$, PQ is minimized.



□

§4.6 Exercises

Exercise 4.48. Let D and E be the trisection points of segment AB, where D is between A and E. Construct a circle using DE as diameter, and let C be a point on the circle. Find the value of

$$\tan \angle ACD \cdot \tan \angle BCE.$$

Hints: 53 33

Exercise 4.49 (Bretschneider's Formula)**.** The area of quadrilateral $ABCD$ is

$$[ABCD] = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2 \frac{\angle A + \angle C}{2}}$$

where a, b, c, d are the sidelengths and $2s = a + b + c + d$. **Hints**: 64 20 40

Exercise 4.50. Given that quadrilateral $ABCD$ has in inscribed circle, show that

$$[ABCD] = \sqrt{abcd}\sin\theta,$$

where a, b, c, d are the side lengths and $\theta = \frac{\angle A + \angle C}{2}$. **Hints**: 42 38

Exercise 4.51. In $\triangle ABC, \angle B = 3\angle C$. If $AB = 10$ and $AC = 15$, compute the length of BC. **Hints**: 103 52

Exercise 4.52. $ARML$ is a convex kite with $A(0,0), R(1,3)$, and $M(7,2)$. Determine the coordinates of L.

Exercise 4.53 (AIME 2001/4)**.** In triangle ABC, angles A and B measure 60 degrees and 45 degrees, respectively. The bisector of angle A intersects $\overline{BC}$ at T, and $AT = 24$. The area of triangle ABC can be written in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$. **Hints**: 80

Exercise 4.54 (AIME I 2003/10)**.** Triangle ABC is isosceles with $AC = BC$ and $\angle ACB = 106°$. Point M is in the interior of the triangle so that $\angle MAC = 7°$ and $\angle MCA = 23°$. Find the number of degrees in $\angle CMB$. **Hints**: 96 99

Exercise 4.55 (AIME 1996/15)**.** In parallelogram $ABCD$, let O be the intersection of diagonals $\overline{AC}$ and $\overline{BD}$. Angles CAB and DBC are each twice as large as angle DBA, and angle ACB is r times as large as angle AOB. Find the greatest integer that does not exceed $1000r$. **Hints**: 58 87 105

§5 3-D Geometry

§5.1 More Vector Geometry

Vectors are very useful, especially for 3D geometry. Consider the distance between a point and a plane. We can find the vector normal to the plane by taking the cross product of two linearly independent vectors lying in the plane. We can then take any vector from a point on the plane to the point of interest and compute its dot product with a unit vector in the direction of the normal. By projecting the arbitrary displacement vector from the plane to the point onto the normal vector, we eliminate the "sideways" portion of the displacement and reduce it to its perpendicular part. The magnitude of the resulting value is the distance we wished to determine.

Theorem 5.1 (Vector on Vector Projection)
Let
$$\text{proj}_{\vec{b}}(\vec{a})$$
be the projection of $\vec{a}$ onto $\vec{b}$. Then
$$\text{proj}_{\vec{b}}(\vec{a}) = a\cos\theta\hat{b},$$
where θ is the angle between the two vectors and $\hat{b}$ is the direction the projection of $\vec{a}$ onto $\vec{b}$ faces (in this case, the direction is the same as $\vec{b}$).

Let us turn to areas now.

Theorem 5.2 (Area-Sine Formula)
Let there exist a triangle ABC such that $BC = a, AC = b$, and $\angle ACB = \theta$. Then the area of $\triangle ABC$ is
$$\frac{1}{2}ab\sin\theta.$$

Notice that this is exactly one half of the expression for the cross product of two vectors in terms of their magnitudes and the angle between them. In the case that the angle involved is not easily determined, such as in a three-dimensional situation, we can directly apply the cross product to vectors representing two sides of the triangle to determine its area. This will eliminate the necessity to find the angle. Similarly, finding the area of parallelogram is simply the cross product of the two vectors that determine it (also note that the area of a parallelogram is simply twice of the triangle).

Now that we have dealt with distances and areas, let us see how we can generalize to volumes. The method is very similar:

Theorem 5.3 (Volume of a Parallelepiped)
A **parallelepiped** (which is basically a shifted box [think 3D parallelogram]) is defined by three vectors $\vec{a}, \vec{b}, \vec{c}$. Then the volume of the parallelepiped is
$$|\vec{a} \times \vec{b}| \cdot \vec{c}.$$
Note that half of this volume is the volume of the tetrahedron defined by $\vec{a}, \vec{b}, \vec{c}$.

Vectors are also great for finding dihedral angles.

Definition 5.4 (Dihedral Angle) — A **dihedral angle** is the angle formed by two intersecting planes.

Definition 5.5 (Normal Vector) — The **normal vector**, often simply called the "normal," to a surface is a vector which is perpendicular to the surface at a given point.

Definition 5.6 (Unit Vector) — A **unit vector** is a vector of magnitude one. We say the unit vector of $\vec{u}$ is $\hat{u}$, and is used to show direction.

Theorem 5.7 (Unit Normal Vector Formula)

Let $\hat{\mathbf{n}}_P$ and $\hat{\mathbf{n}}_Q$ be the unit normal vectors of planes P and Q, respectively. Also, let $\vec{p_1}$ and $\vec{p_2}$ be vectors in the plane P and let $\vec{q_1}$ and $\vec{q_2}$ be vectors in the plane Q. Then

$$\hat{\mathbf{n}}_P = \frac{\vec{p_1} \times \vec{p_2}}{p_1 p_2},$$

and

$$\hat{\mathbf{n}}_Q = \frac{\vec{q_1} \times \vec{q_2}}{q_1 q_2}.$$

Theorem 5.8 (Dihedral Angle Formula)

Let θ be the angle between two planes P and Q, and let $\hat{\mathbf{n}}_P$ and $\hat{\mathbf{n}}_Q$ be the unit normal vectors of P and Q, respectively. Then

$$\cos\theta = \hat{\mathbf{n}}_P \cdot \hat{\mathbf{n}}_Q.$$

We can try our hand at the following example:

Example 5.9 (AIME II 2016/14)

Equilateral $\triangle ABC$ has side length 600. Points P and Q lie outside the plane of $\triangle ABC$ and are on opposite sides of the plane. Furthermore, $PA = PB = PC$, and $QA = QB = QC$, and the planes of $\triangle PAB$ and $\triangle QAB$ form a 120° dihedral angle (the angle between the two planes). There is a point O whose distance from each of A, B, C, P, and Q is d. Find d.

,

§5.2 Exercises

Exercise 5.10. Let PQ be the line passing through the points $P = (-1, 0, 3)$ and $Q = (0, -2, -1)$. Determine the shortest distance from PQ to the origin. **Hints**: 1

Exercise 5.11. A parallelpiped has a vertex at $(1, 2, 3)$, and adjacent vertices (that form edges with this vertex) at $(3, 5, 7), (1, 6, -2)$, and $(6, 3, 6)$. Find the volume of this parallelpiped. **Hints**: 57

§6 Trigonometric Substitution

Trigonometry substitution is extremely useful for a variety of problems. Here are a few substitutions to employ.

Theorem 6.1 (Weierstrauss Substitution)
Let $t = \tan \frac{x}{2}$, where $x \in (-\pi, \pi)$. Then

$$\sin \frac{x}{2} = \frac{t}{\sqrt{1+t^2}},$$

and

$$\cos \frac{x}{2} + \frac{1}{\sqrt{1+t^2}}.$$

Similarly,

$$\sin x = \frac{2t}{1+t^2},$$

$$\cos x = \frac{1-t^2}{1+t^2},$$

and

$$\tan x = \frac{2t}{1-t^2}.$$

Theorem 6.2 (Trigonometric Triangle-Angle Condition)
Let α, β, γ be angles in the range $(0, \pi)$. Then α, β, γ are angles of a triangle if and only if

$$\tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} =,$$

or

$$\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} + 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = 1.$$

The former is useful for expressions of the form $ab + bc + ca = 1$.

Theorem 6.3 (Triangle-Angle Substitution)
Let α, β, γ be angles of a triangle. Then

$$A = \frac{\pi - \alpha}{2}, B = \frac{\pi - \beta}{2}, C = \frac{\pi - \gamma}{2}$$

transforms the triangle into an acute triangle with angles A, B, C.

Theorem 6.4 ($ab + bc + ca = 1$ Substitution)
Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Then we can substitute

$$a = \frac{\tan\alpha}{2}, b = \frac{\tan\beta}{2}, c = \frac{\tan\gamma}{2},$$

or

$$a = \cot A, b = \cot B, c = \cot C,$$

where α, β, γ and A, B, C are angles of a triangle.

Theorem 6.5 ($a + b + c = abc$ Substitution)
Let a, b, c be positive real numbers such that $a + b + c = abc$. Then we can substitute

$$a = \cot\frac{\alpha}{2}, b = \cot\frac{\beta}{2}, c = \cot\frac{\gamma}{2},$$

or

$$a = \tan A, b = \tan B, c = \tan C,$$

where α, β, γ are angles of a triangle.

Theorem 6.6 ($a^2 + b^2 + c^2 + 2abc = 1$ Substitution)
Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + 2abc = 1$. Then we can substitute

$$a = \sin\frac{\alpha}{2}, b = \sin\frac{\beta}{2}, c = \sin\frac{\gamma}{2},$$

or

$$a = \cos A, b = \cos B, c = \cos C.$$

Example 6.7 (Darij Grinberg)
Let x, y, z be positive real numbers. Prove that

$$\sqrt{x(y+z)} + \sqrt{y(z+x)} + \sqrt{z(x+y)} \geq 2\sqrt{\frac{(x+y)(y+z)(z+x)}{x+y+z}}.$$

Solution. We can rewrite this inequality as

$$\sum_{\text{cyc}} \sqrt{\frac{x(x+y+z)}{(x+y)(y+z)}} \geq 2.$$

These values can be substituted for $\sin A, \sin B$, and $\sin C$, so it suffices to prove

$$\sin A + \sin B + \sin C \geq 2,$$

where A, B, C are angles of an acute triangle (prove why this substitution is true!). Using Jordan's Inequality, we have

$$\frac{2\alpha}{\pi} \leq \sin\alpha \leq \alpha,$$

and summing cyclically gives us the desired result. □

Example 6.8 (HMMT)
Find the minimum possible value of $\sqrt{58-42x}+\sqrt{149-140\sqrt{1-x^2}}$ where $-1 \le x \le 1$.

Solution. The $\sqrt{1-x^2}$ is an obvious indicator of trigonometric substiution. Thus, if we let

$$x = \cos\theta,$$

then

$$\sqrt{1-x^2} = \sin\theta.$$

While $\sqrt{58-42x}$ is rather innocent, 149 and 140 should indicate Law of Cosines. In particular,

$$149 = 7^2 + 10^2,$$

$$140 = 2 \cdot 7 \cdot 10.$$

If we turn our attention to 58 and 42, we have

$$58 = 3^2 + 7^2,$$

$$42 = 2 \cdot 3 \cdot 7.$$

Thus, if we have a triangle with side lengths 3 and 7, with angle θ between them, then $\sqrt{58-42x}$ would be the last side. Similarly, if we have a triangle with side lengths 7 and 10, with angle $90^\circ - \theta$ between them, $\sqrt{149-140\sqrt{1-x^2}}$ would be the last side. The θ and $90^\circ-\theta$, paired with the common 7, inspires us to combine these two triangles such that the angles of measure θ and $90^\circ-\theta$ become 90°, and the two sides of length 7 become one side. Thus, we have a triangle with side lengths $3, 10$, and $\sqrt{58-42x}+\sqrt{149-140\sqrt{1-x^2}}$, with a 90-degree angle between 3 and 10. Thus,

$$\sqrt{58-42x}+\sqrt{149-140\sqrt{1-x^2}} \ge \sqrt{3^2+10^2} = \boxed{\sqrt{109}}.$$

□

Theorem 6.9 (Trigonometric Inequalities)
Let A, B, C be angles of triangle ABC. Then

1. $\cos A + \cos B + \cos C \le \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \le \frac{3}{2}$
2. $\sin A + \sin B + \sin C \le \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \le \frac{3\sqrt{3}}{2}$
3. $\cos A \cos B \cos C \le \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} \le \frac{1}{8}$
4. $\sin A \sin B \sin C \le \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2} \le \frac{3\sqrt{3}}{8}$
5. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} \ge 3\sqrt{3}$
6. $\cos^2 A + \cos^2 B + \cos^2 C \ge \sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} \ge \frac{3}{4}$
7. $\sin^2 A + \sin^2 B + \sin^2 C \le \cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} \le \frac{9}{4}$
8. $\cot A + \cot B + \cot C \ge \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \ge \sqrt{3}$

Theorem 6.10 (Well-Known Triangle Trigonometric Identities)
Let A, B, C be angles of triangle ABC. Then

1. $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
2. $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
3. $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$
4. $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A\cos B\cos C$

Theorem 6.11 (Well-Known Trigonometric Identities)
For arbitrary angles α, β, γ,

$$\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2},$$

and

$$\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\alpha}{2}\cos\frac{\gamma+\alpha}{2}.$$

§7 Worked Through Problems

Example 7.1 (AIME 1989/10)
Let a, b, c be the three sides of a triangle, and let α, β, γ, be the angles opposite them. If $a^2 + b^2 = 1989c^2$, find

$$\frac{\cot\gamma}{\cot\alpha + \cot\beta}.$$

Solution. We can draw the altitude h to c, to get two right triangles. $\cot\alpha + \cot\beta = \frac{c}{h}$, from the definition of the cotangent. From the definition of area, $h = \frac{2A}{c}$, so $\cot\alpha + \cot\beta = \frac{c^2}{2A}$.

Now we evaluate the numerator:

$$\cot\gamma = \frac{\cos\gamma}{\sin\gamma}$$

From the Law of Cosines and the sine area formula,

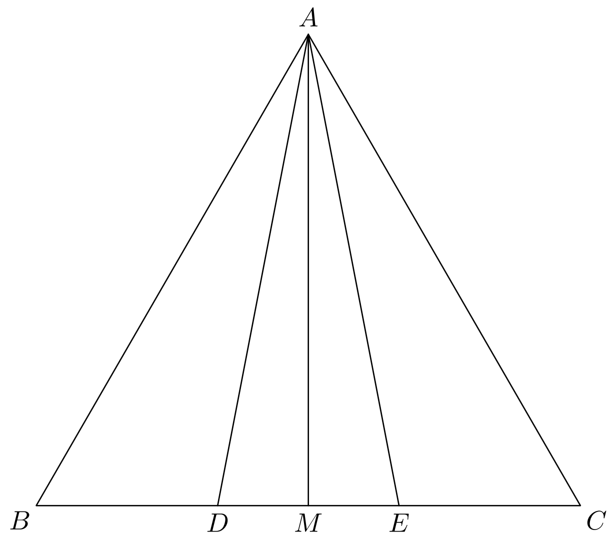
$$\cos\gamma = \frac{1988c^2}{2ab}$$
$$\sin\gamma = \frac{2A}{ab}$$
$$\cot\gamma = \frac{\cos\gamma}{\sin\gamma} = \frac{1988c^2}{4A}$$

Then $\frac{\cot\gamma}{\cot\alpha+\cot\beta} = \frac{\frac{1988c^2}{4A}}{\frac{c^2}{2A}} = \frac{1988}{2} = \boxed{994}$. □

Example 7.2 (AIME II 2013/5)

In equilateral $\triangle ABC$ let points D and E trisect $\overline{BC}$. Then $\sin(\angle DAE)$ can be expressed in the form $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is an integer that is not divisible by the square of any prime. Find $a+b+c$.

Solution. Without loss of generality, assume the triangle sides have length 3. Then the trisected side is partitioned into segments of length 1, making your computation easier.



Let M be the midpoint of $\overline{DE}$. Then ΔMCA is a 30-60-90 triangle with $MC = \dfrac{3}{2}$, $AC = 3$ and $AM = \dfrac{3\sqrt{3}}{2}$. Since the triangle ΔAME is right, then we can find the length of $\overline{AE}$ by pythagorean theorem, $AE = \sqrt{7}$. Therefore, since ΔAME is a right triangle, we can easily find $\sin(\angle EAM) = \dfrac{1}{2\sqrt{7}}$ and $\cos(\angle EAM) = \sqrt{1-\sin(\angle EAM)^2} = \dfrac{3\sqrt{3}}{2\sqrt{7}}$. So we can use the double angle formula for sine, $\sin(\angle EAD) = 2\sin(\angle EAM)\cos(\angle EAM) = \dfrac{3\sqrt{3}}{14}$. Therefore, $a+b+c = \boxed{020}$. □

Example 7.3 (AIME 1994/10)

In triangle ABC, angle C is a right angle and the altitude from C, meets $\overline{AB}$, at D. The lengths of the sides of $\triangle ABC$, are integers, $BD = 29^3$, , and $\cos B = m/n$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. We will solve for $\cos B$ using $\triangle CBD$, which gives us $\cos B = \frac{29^3}{BC}$. By the Pythagorean Theorem on $\triangle CBD$, we have $BC^2 - DC^2 = (BC+DC)(BC-DC) = 29^6$. Trying out factors of 29^6, we can either guess and check or just guess to find that $BC + DC = 29^4$ and $BC - DC = 29^2$ (The other pairs give answers over 999). Adding these, we have $2BC = 29^4 + 29^2$ and $\frac{29^3}{BC} = \frac{2*29^3}{29^2(29^2+1)} = \frac{58}{842} = \frac{29}{421}$, and our answer is $\boxed{450}$. □

Example 7.4 (AIME 1996/10)

Find the smallest positive integer solution to $\tan 19x^\circ = \dfrac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ}$.

Solution. Note that

$$\begin{aligned}&\frac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ}\\ &= \frac{\sin 186^\circ + \sin 96^\circ}{\sin 186^\circ - \sin 96^\circ}\\ &= \frac{\sin(141^\circ + 45^\circ) + \sin(141^\circ - 45^\circ)}{\sin(141^\circ + 45^\circ) - \sin(141^\circ - 45^\circ)}\\ &= \frac{2\sin 141^\circ \cos 45^\circ}{2\cos 141^\circ \sin 45^\circ} = \tan 141^\circ.\end{aligned}$$

The period of the tangent function is 180°, and the tangent function is one-to-one over each period of its domain.

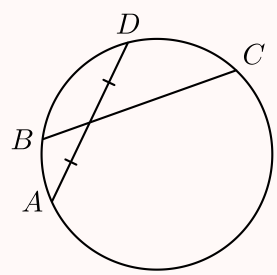
Thus, $19x \equiv 141 \pmod{180}$.

Since $19^2 \equiv 361 \equiv 1 \pmod{180}$, multiplying both sides by 19 yields $x \equiv 141 \cdot 19 \equiv (140+1)(18+1) \equiv 0 + 140 + 18 + 1 \equiv 159 \pmod{180}$.

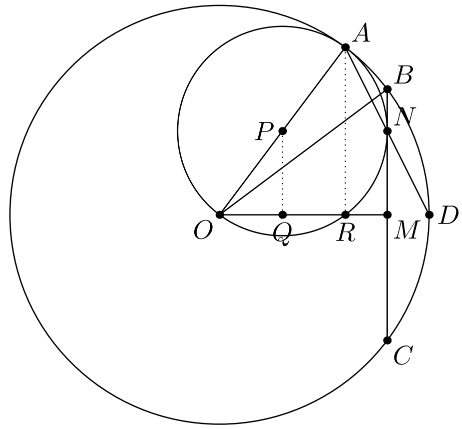
Therefore, the smallest positive solution is $x = \boxed{159}$. □

Example 7.5 (AIME 1983/15)

The adjoining figure shows two intersecting chords in a circle, with B on minor arc AD. Suppose that the radius of the circle is 5, that $BC = 6$, and that AD is bisected by BC. Suppose further that AD is the only chord starting at A which is bisected by BC. It follows that the sine of the central angle of minor arc AB is a rational number. If this number is expressed as a fraction $\frac{m}{n}$ in lowest terms, what is the product mn?



Solution. (Figure by AoPS User Adamz.)



Let A be any fixed point on circle O, and let AD be a chord of circle O. The locus of midpoints N of the chord AD is a circle P, with diameter AO. Generally, the circle P can intersect the chord BC at two points, one point, or they may not have a point of intersection. By the problem condition, however, the circle P is tangent to BC at point N.

Let M be the midpoint of the chord BC. From right triangle OMB, we have $OM = \sqrt{OB^2 - BM^2} = 4$. This gives $\tan\angle BOM = \frac{BM}{OM} = \frac{3}{4}$.

Notice that the distance OM equals $PN + PO\cos\angle AOM = r(1+\cos\angle AOM)$, where r is the radius of circle P.

Hence

$$\cos\angle AOM = \frac{OM}{r} - 1 = \frac{2OM}{R} - 1 = \frac{8}{5} - 1 = \frac{3}{5}$$

(where R represents the radius, 5, of the large circle given in the question). Therefore, since $\angle AOM$ is clearly acute, we see that

$$\tan\angle AOM = \frac{\sqrt{1-\cos^2\angle AOM}}{\cos\angle AOM} = \frac{\sqrt{5^2-3^2}}{3} = \frac{4}{3}$$

Next, notice that $\angle AOB = \angle AOM - \angle BOM$. We can therefore apply the subtraction formula for tan to obtain

$$\tan\angle AOB = \frac{\tan\angle AOM - \tan\angle BOM}{1+\tan\angle AOM\cdot\tan\angle BOM} = \frac{\frac{4}{3}-\frac{3}{4}}{1+\frac{4}{3}\cdot\frac{3}{4}} = \frac{7}{24}$$

It follows that $\sin\angle AOB = \frac{7}{\sqrt{7^2+24^2}} = \frac{7}{25}$, such that the answer is $7\cdot 25 = \boxed{175}$. □

Example 7.6 (AIME I 2003/11)

An angle x is chosen at random from the interval $0^\circ < x < 90^\circ$. Let p be the probability that the numbers $\sin^2 x, \cos^2 x$, and $\sin x\cos x$ are not the lengths of the sides of a triangle. Given that $p = d/n$, where d is the number of degrees in $\arctan m$ and m and n are positive integers with $m+n < 1000$, find $m+n$.

Solution. Note that the three expressions are symmetric with respect to interchanging sin and cos, and so the probability is symmetric around 45°. Thus, take $0 < x < 45$ so that $\sin x < \cos x$. Then $\cos^2 x$ is the largest of the three given expressions and those three lengths not forming a triangle is equivalent to a violation of the triangle inequality

$$\cos^2 x > \sin^2 x + \sin x\cos x$$

This is equivalent to

$$\cos^2 x - \sin^2 x > \sin x\cos x$$

and, using some of our trigonometric identities, we can re-write this as $\cos 2x > \frac{1}{2}\sin 2x$. Since we've chosen $x \in (0, 45)$, $\cos 2x > 0$ so

$$2 > \tan 2x \Longrightarrow x < \frac{1}{2}\arctan 2.$$

The probability that x lies in this range is $\frac{1}{45}\cdot\left(\frac{1}{2}\arctan 2\right) = \frac{\arctan 2}{90}$ so that $m = 2$, $n = 90$ and our answer is $\boxed{092}$. □

Example 7.7 (AIME I 2003/12)

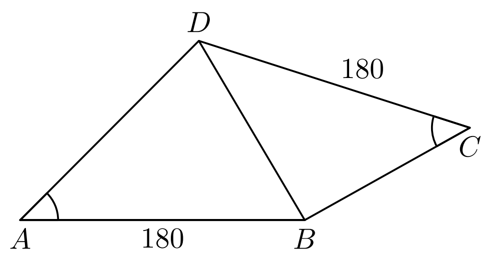
In convex quadrilateral $ABCD, \angle A \cong \angle C, AB = CD = 180$, and $AD \neq BC$. The perimeter of $ABCD$ is 640. Find $\lfloor 1000\cos A\rfloor$. (The notation $\lfloor x\rfloor$ means the greatest integer that is less than or equal to x.)

Solution. By the Law of Cosines on $\triangle ABD$ at angle A and on $\triangle BCD$ at angle C (note $\angle C = \angle A$),

$$180^2 + AD^2 - 360 \cdot AD \cos A = 180^2 + BC^2 - 360 \cdot BC \cos A$$
$$(AD^2 - BC^2) = 360(AD - BC) \cos A$$
$$(AD - BC)(AD + BC) = 360(AD - BC) \cos A$$
$$(AD + BC) = 360 \cos A$$

We know that $AD + BC = 640 - 360 = 280$. $\cos A = \dfrac{280}{360} = \dfrac{7}{9} = 0.777\ldots$

$\lfloor 1000 \cos A \rfloor = \boxed{777}$.



□

Example 7.8 (AIME I 2014/10)

A disk with radius 1 is externally tangent to a disk with radius 5. Let A be the point where the disks are tangent, C be the center of the smaller disk, and E be the center of the larger disk. While the larger disk remains fixed, the smaller disk is allowed to roll along the outside of the larger disk until the smaller disk has turned through an angle of 360°. That is, if the center of the smaller disk has moved to the point D, and the point on the smaller disk that began at A has now moved to point B, then $\overline{AC}$ is parallel to $\overline{BD}$. Then $\sin^2(\angle BEA) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

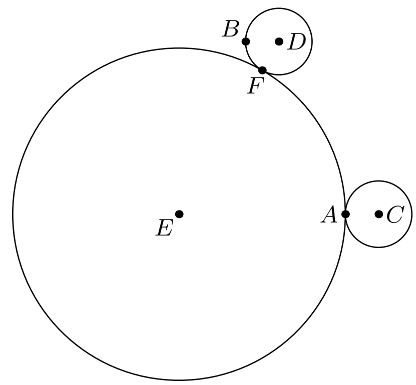
Solution. First, we determine how far the small circle goes. For the small circle to rotate completely around the circumference, it must rotate 5 times (the circumference of the small circle is 2π while the larger one has a circumference of 10π) plus the extra rotation the circle gets for rotating around the circle, for a total of 6 times. Therefore, one rotation will bring point D 60° from C.

Now, draw $\triangle DBE$, and call $\angle BED$ x, in degrees. We know that $\overline{ED}$ is 6, and $\overline{BD}$ is 1. Since $EC || BD$, $\angle BDE = 60^\circ$. By the Law of Cosines, $\overline{BE}^2 = 36 + 1 - 2 \times 6 \times 1 \times \cos 60^\circ = 36 + 1 - 6 = 31$, and since lengths are positive, $\overline{BE} = \sqrt{31}$.

By the Law of Sines, we know that $\frac{1}{\sin x} = \frac{\sqrt{31}}{\sin 60^\circ}$, so $\sin x = \frac{\sin 60^\circ}{\sqrt{31}} = \frac{\sqrt{93}}{62}$. As x is clearly between 0 and 90°, $\cos x$ is positive. As $\cos x = \sqrt{1 - \sin^2 x}$, $\cos x = \frac{11\sqrt{31}}{62}$.

Now we use the angle sum formula to find the sine of $\angle BEA$: $\sin 60^\circ \cos x + \cos 60^\circ \sin x = \frac{\sqrt{3}}{2}\frac{11\sqrt{31}}{62} + \frac{1}{2}\frac{\sqrt{93}}{62} = \frac{11\sqrt{93}+\sqrt{93}}{124} = \frac{12\sqrt{93}}{124} = \frac{3\sqrt{93}}{31} = \frac{3\sqrt{31}\sqrt{3}}{31} = \frac{3\sqrt{3}}{\sqrt{31}}$.

Finally, we square this to get $\frac{9\times 3}{31} = \frac{27}{31}$, so our answer is $27 + 31 = \boxed{058}$.



$\square$

Example 7.9 (AIME 1997/14)

Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Let $\frac{m}{n}$ be the probability that $\sqrt{2+\sqrt{3}} \le |v+w|$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. We know that

$$z^{1997} = 1 = 1(\cos 0 + i \sin 0).$$

By De Moivre's Theorem, we find that ($k \in \{0, 1, \ldots, 1996\}$)

$$z = \cos\left(\frac{2\pi k}{1997}\right) + i \sin\left(\frac{2\pi k}{1997}\right).$$

Now, let v be the root corresponding to $\theta = \frac{2\pi m}{1997}$, and let w be the root corresponding to $\theta = \frac{2\pi n}{1997}$. The magnitude of $v+w$ is therefore:

$$\sqrt{\left(\cos\left(\frac{2\pi m}{1997}\right) + \cos\left(\frac{2\pi n}{1997}\right)\right)^2 + \left(\sin\left(\frac{2\pi m}{1997}\right) + \sin\left(\frac{2\pi n}{1997}\right)\right)^2}$$

$$= \sqrt{2 + 2\cos\left(\frac{2\pi m}{1997}\right)\cos\left(\frac{2\pi n}{1997}\right) + 2\sin\left(\frac{2\pi m}{1997}\right)\sin\left(\frac{2\pi n}{1997}\right)}$$

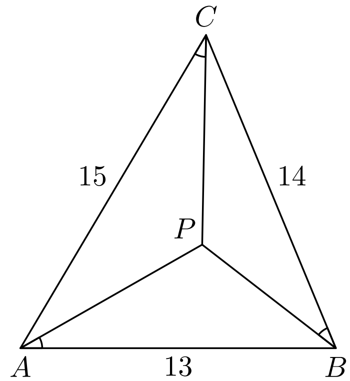
We need $\cos\left(\frac{2\pi m}{1997}\right)\cos\left(\frac{2\pi n}{1997}\right) + \sin\left(\frac{2\pi m}{1997}\right)\sin\left(\frac{2\pi n}{1997}\right) \ge \frac{\sqrt{3}}{2}$. The cosine difference identity simplifies that to $\cos\left(\frac{2\pi m}{1997} - \frac{2\pi n}{1997}\right) \ge \frac{\sqrt{3}}{2}$. Thus, $|m-n| \le \frac{\pi}{6} \cdot \frac{1997}{2\pi} = \lfloor \frac{1997}{12} \rfloor = 166$.

Therefore, m and n cannot be more than 166 away from each other. This means that for a given value of m, there are 332 values for n that satisfy the inequality; 166 of them $> m$, and 166 of them $< m$. Since m and n must be distinct, n can have 1996 possible values. Therefore, the probability is $\frac{332}{1996} = \frac{83}{499}$. The answer is then $499 + 83 = \boxed{582}$. $\square$

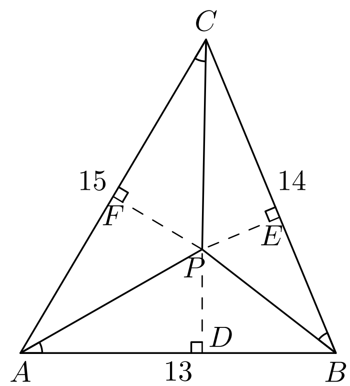
Example 7.10 (AIME 1999/14)

Point P is located inside triangle ABC so that angles PAB, PBC, and PCA are all congruent. The sides of the triangle have lengths $AB = 13, BC = 14$, and $CA = 15$, and the tangent of angle PAB is m/n, where m and n are relatively prime positive integers. Find $m+n$.

Solution. The following is the figure for this problem.



Drop perpendiculars from P to the three sides of $\triangle ABC$ and let them meet $\overline{AB}, \overline{BC}$, and $\overline{CA}$ at D, E, and F respectively.



Let $BE = x, CF = y$, and $AD = z$. We have that

$$\begin{aligned} DP &= z \tan\theta \\ EP &= x \tan\theta \\ FP &= y \tan\theta \end{aligned}$$

We can then use the tool of calculating area in two ways

$$\begin{aligned} [ABC] &= [PAB] + [PBC] + [PCA] \\ &= \frac{1}{2}(13)(z\tan\theta) + \frac{1}{2}(14)(x\tan\theta) + \frac{1}{2}(15)(y\tan\theta) \\ &= \frac{1}{2}\tan\theta(13z + 14x + 15y) \end{aligned}$$

On the other hand,

$$\begin{aligned} [ABC] &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{21 \cdot 6 \cdot 7 \cdot 8} \\ &= 84 \end{aligned}$$

We still need $13z + 14x + 15y$ though. We have all these right triangles and we haven't even touched Pythagoras. So we give it a shot:

$$x^2 + x^2\tan^2\theta = z^2\tan^2\theta + (13 - z)^2 \tag{1}$$
$$z^2 + z^2\tan^2\theta = y^2\tan^2\theta + (15 - y)^2 \tag{2}$$
$$y^2 + y^2\tan^2\theta = x^2\tan^2\theta + (14 - x)^2 \tag{3}$$

Adding $(1)+(2)+(3)$ gives

$$\begin{aligned} x^2+y^2+z^2 &= (14-x)^2+(15-y)^2+(13-z)^2 \\ \Rightarrow 13z+14x+15y &= 295 \end{aligned}$$

Recall that we found that $[ABC]=\frac{1}{2}\tan\theta(13z+14x+15y)=84$. Plugging in $13z+14x+15y=295$, we get $\tan\theta=\frac{168}{295}$, giving us $\boxed{463}$ for an answer. □

Example 7.11 (AIME I 2007/12)

In isosceles triangle $\triangle ABC$, A is located at the origin and B is located at $(20,0)$. Point C is in the first quadrant with $AC=BC$ and angle $BAC=75^\circ$. If triangle ABC is rotated counterclockwise about point A until the image of C lies on the positive y-axis, the area of the region common to the original and the rotated triangle is in the form $p\sqrt{2}+q\sqrt{3}+r\sqrt{6}+s$, where p,q,r,s are integers. Find $\frac{p-q+r-s}{2}$.

Solution. Let the new triangle be $\triangle AB'C'$ (A, the origin, is a vertex of both triangles). Let $\overline{B'C'}$ intersect with $\overline{AC}$ at point D, $\overline{BC}$ intersect with $\overline{B'C'}$ at E, and $\overline{BC}$ intersect with $\overline{AB'}$ at F. The region common to both triangles is the quadrilateral $ADEF$. Notice that $[ADEF]=[\triangle ADB']-[\triangle EFB']$, where we let $[\dots]$ denote area.

To find $[\triangle ADB']$: Since $\angle B'AC'$ and $\angle BAC$ both have measures 75°, both of their complements are 15°, and $\angle DAB'=90-2(15)=60^\circ$. We know that $\angle DB'A=75^\circ$, so $\angle ADB'=180-60-75=45^\circ$.

Thus $\triangle ADB'$ is a $45-60-75\triangle$. It can be solved by drawing an altitude splitting the 75° angle into 30° and 45° angles, forming a $30-60-90$ right triangle and a $45-45-90$ isosceles right triangle. Since we know that $AB'=20$, the base of the $30-60-90$ triangle is 10, the base of the $45-45-90$ is $10\sqrt{3}$, and their common height is $10\sqrt{3}$. Thus, the total area of $[\triangle ADB']=\frac{1}{2}(10\sqrt{3})(10\sqrt{3}+10)=\boxed{150+50\sqrt{3}}$.

To find $[\triangle EFB']$: Since $\triangle AFB$ is also a $15-75-90$ triangle,

$AF=20\sin 75=20\sin(30+45)=20\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)=5\sqrt{2}+5\sqrt{6}$ and

$FB'=AB'-AF=20-5\sqrt{2}-5\sqrt{6}$ Since $[\triangle EFB']=\frac{1}{2}(FB'\cdot EF)=\frac{1}{2}(FB')(FB'\tan 75^\circ)$. With some horrendous algebra, we can calculate

$$\begin{aligned} [\triangle EFB'] &= \frac{1}{2}\tan(30+45)\cdot(20-5\sqrt{2}-5\sqrt{6})^2 \\ &= 25\left(\frac{\frac{1}{\sqrt{3}}+1}{1-\frac{1}{\sqrt{3}}}\right)\left(8-2\sqrt{2}-2\sqrt{6}-2\sqrt{2}+1+\sqrt{3}-2\sqrt{6}+\sqrt{3}+3\right) \\ &= 25(2+\sqrt{3})(12-4\sqrt{2}-4\sqrt{6}+2\sqrt{3}) \\ [\triangle EFB'] &= \boxed{-500\sqrt{2}+400\sqrt{3}-300\sqrt{6}+750}. \end{aligned}$$

To finish,

$$\begin{aligned} [ADEF] &= [\triangle ADB']-[\triangle EFB'] \\ &= \left(150+50\sqrt{3}\right)-\left(-500\sqrt{2}+400\sqrt{3}-300\sqrt{6}+750\right) \\ &= 500\sqrt{2}-350\sqrt{3}+300\sqrt{6}-600 \end{aligned}$$

Hence, $\frac{p-q+r-s}{2}=\frac{500+350+300+600}{2}=\frac{1750}{2}=\boxed{875}$. □

Example 7.12 (AIME I 2012/12)

Let $\triangle ABC$ be a right triangle with right angle at C. Let D and E be points on $\overline{AB}$ with D between A and E such that $\overline{CD}$ and $\overline{CE}$ trisect $\angle C$. If $\frac{DE}{BE} = \frac{8}{15}$, then $\tan B$ can be written as $\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers, and p is a positive integer not divisible by the square of any prime. Find $m+n+p$.

Solution. Without loss of generality, set $CB = 1$. Then, by the Angle Bisector Theorem on triangle DCB, we have $CD = \frac{8}{15}$. We apply the Law of Cosines to triangle DCB to get $1 + \frac{64}{225} - \frac{8}{15} = BD^2$, which we can simplify to get $BD = \frac{13}{15}$.

Now, we have $\cos\angle B = \frac{1+\frac{169}{225}-\frac{64}{225}}{\frac{26}{15}}$ by another application of the Law of Cosines to triangle DCB, so $\cos\angle B = \frac{11}{13}$. In addition, $\sin\angle B = \sqrt{1 - \frac{121}{169}} = \frac{4\sqrt{3}}{13}$, so $\tan\angle B = \frac{4\sqrt{3}}{11}$.

Our final answer is $4 + 3 + 11 = \boxed{018}$. $\square$

Example 7.13 (AIME II 2014/12)

Suppose that the angles of $\triangle ABC$ satisfy $\cos(3A) + \cos(3B) + \cos(3C) = 1$. Two sides of the triangle have lengths 10 and 13. There is a positive integer m so that the maximum possible length for the remaining side of $\triangle ABC$ is $\sqrt{m}$. Find m.

Solution. Note that $\cos 3C = -\cos(3A + 3B)$. Thus, our expression is of the form $\cos 3A + \cos 3B - \cos(3A + 3B) = 1$. Let $\cos 3A = x$ and $\cos 3B = y$.

Using the fact that $\cos(3A + 3B) = \cos 3A \cos 3B - \sin 3A \sin 3B = xy - \sqrt{1-x^2}\sqrt{1-y^2}$, we get $x + y - xy + \sqrt{1-x^2}\sqrt{1-y^2} = 1$, or $\sqrt{1-x^2}\sqrt{1-y^2} = xy - x - y + 1 = (x-1)(y-1)$.

Squaring both sides, we get $(1-x^2)(1-y^2) = [(x-1)(y-1)]^2$. Cancelling factors, $(1+x)(1+y) = (1-x)(1-y)$.

Notice here that we cancelled out one factor of (x-1) and (y-1), which implies that (x-1) and (y-1) were not 0. If indeed they were 0 though, we would have $\cos(3A) - 1 = 0, \cos(3A) = 1$

For this we could say that A must be 120 degrees for this to work. This is one case. The B case follows in the same way, where B must be equal to 120 degrees. This doesn't change the overall solution though, as then the other angles are irrelevant (this is the largest angle, implying that this will have the longest side and so we would want to have the 120 degreee angle opposite of the unknown side).

Expanding, $1 + x + y + xy = 1 - x - y + xy \to x + y = -x - y$.

Simplification leads to $x + y = 0$.

Therefore, $\cos(3C) = 1$. So $\angle C$ could be 0° or 120°. We eliminate 0° and use law of cosines to get our answer:

$$m = 10^2 + 13^2 - 2 \cdot 10 \cdot 13 \cos\angle C$$

$$\to m = 269 - 260 \cos 120^\circ = 269 - 260\left(-\frac{1}{2}\right)$$

$$\to m = 269 + 130 = \boxed{399}.$$

$\square$

Example 7.14 (AIME I 2011/14)

Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a regular octagon. Let M_1, M_3, M_5, and M_7 be the midpoints of sides $\overline{A_1A_2}$, $\overline{A_3A_4}$, $\overline{A_5A_6}$, and $\overline{A_7A_8}$, respectively. For $i = 1, 3, 5, 7$, ray R_i is constructed from M_i towards the interior of the octagon such that $R_1 \perp R_3$, $R_3 \perp R_5$, $R_5 \perp R_7$, and $R_7 \perp R_1$. Pairs of rays R_1 and R_3, R_3 and R_5, R_5 and R_7, and R_7 and R_1 meet at B_1, B_3, B_5, B_7 respectively. If $B_1B_3 = A_1A_2$, then $\cos 2\angle A_3M_3B_1$ can be written in the form $m - \sqrt{n}$, where m and n are positive integers. Find $m + n$.

Solution. Let $\theta = \angle M_1M_3B_1$. Thus we have that $\cos 2\angle A_3M_3B_1 = \cos\left(2\theta + \frac{\pi}{2}\right) = -\sin 2\theta$.

Since $A_1A_2A_3A_4A_5A_6A_7A_8$ is a regular octagon and $B_1B_3 = A_1A_2$, let $k = A_1A_2 = A_2A_3 = B_1B_3$.

Extend $\overline{A_1A_2}$ and $\overline{A_3A_4}$ until they intersect. Denote their intersection as I_1. Through similar triangles and the $45 - 45 - 90$ triangles formed, we find that $M_1M_3 = \frac{k}{2}(2 + \sqrt{2})$.

We also have that$\triangle M_7B_7M_1 = \triangle M_1B_1M_3$ through ASA congruence ($\angle B_7M_7M_1 = \angle B_1M_1M_3$, $M_7M_1 = M_1M_3$, $\angle B_7M_1M_7 = \angle B_1M_3M_1$). Therefore, we may let $n = M_1B_7 = M_3B_1$.

Thus, we have that $\sin\theta = \frac{n+k}{\frac{k}{2}(2+\sqrt{2})}$ and that $\cos\theta = \frac{n}{\frac{k}{2}(2+\sqrt{2})}$. Therefore $\sin\theta - \cos\theta = \frac{k}{\frac{k}{2}(2+\sqrt{2})} = \frac{2}{2+\sqrt{2}} = 2 - \sqrt{2}$.

Squaring gives that $\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta = 6 - 4\sqrt{2}$ and consequently that $-2\sin\theta\cos\theta = 5 - 4\sqrt{2} = -\sin 2\theta$ through the identities $\sin^2\theta + \cos^2\theta = 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

Thus we have that $\cos 2\angle A_3M_3B_1 = 5 - 4\sqrt{2} = 5 - \sqrt{32}$. Therefore $m + n = 5 + 32 = \boxed{037}$. □

Example 7.15 (AIME II 2013/15)

Let A, B, C be angles of an acute triangle with

$$\cos^2 A + \cos^2 B + 2\sin A\sin B\cos C = \frac{15}{8} \text{ and}$$
$$\cos^2 B + \cos^2 C + 2\sin B\sin C\cos A = \frac{14}{9}$$

There are positive integers p, q, r, and s for which

$$\cos^2 C + \cos^2 A + 2\sin C\sin A\cos B = \frac{p - q\sqrt{r}}{s},$$

where $p + q$ and s are relatively prime and r is not divisible by the square of any prime. Find $p + q + r + s$.

Solution. Let's draw the triangle. Since the problem only deals with angles, we can go ahead and set one of the sides to a convenient value. Let $BC = \sin A$.

By the Law of Sines, we must have $CA = \sin B$ and $AB = \sin C$.

Now let us analyze the given:

$$\begin{aligned}\cos^2 A + \cos^2 B + 2\sin A\sin B\cos C &= 1 - \sin^2 A + 1 - \sin^2 B + 2\sin A\sin B\cos C \\ &= 2 - (\sin^2 A + \sin^2 B - 2\sin A\sin B\cos C)\end{aligned}$$

Now we can use the Law of Cosines to simplify this:

$$= 2 - \sin^2 C$$

Therefore:

$$\sin C = \sqrt{\frac{1}{8}}, \cos C = \sqrt{\frac{7}{8}}.$$

Similarly,

$$\sin A = \sqrt{\frac{4}{9}}, \cos A = \sqrt{\frac{5}{9}}.$$

Note that the desired value is equivalent to $2 - \sin^2 B$, which is $2 - \sin^2(A+C)$. All that remains is to use the sine addition formula and, after a few minor computations, we obtain a result of $\dfrac{111 - 4\sqrt{35}}{72}$. Thus, the answer is $111 + 4 + 35 + 72 = \boxed{222}$.

Note that the problem has a flaw because $\cos B < 0$ which contradicts with the statement that it's an acute triangle. Would be more accurate to state that A and C are smaller than 90. Also note that the identity $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$ would have easily solved the problem. □

§8 Parting Words and Final Problems

So with this, you should be able to solve almost any AIME Problem on trigonometry and its applications. We hope this document helped you learn a bit about how to use trigonometry in all kinds of contexts, even ones that aren't obviously apparent. In addition, we hope that this will boost your geometry skills, as trigonometry is very commonly used to solve problems. Any suggestion would be extremely helpful, whether it would be problem suggestions, mistakes we made, or stuff we should explain better. Here's a final problem set that should incorporate (almost) every AIME Problem which requires trigonometry (that hasn't been solved above):

Problem 8.1. Evaluate $\sin(\frac{7\pi}{6})$. **Hints**: 104

Problem 8.2. Evaluate $\tan(\frac{-3\pi}{4})$. **Hints**: 4

Problem 8.3. Solve $\sin(x) + \cos(x) = 0$ for x. **Hints**: 7

Problem 8.4. Solve $2\cos(2x) + 1 = 0$ for x. **Hints**: 34

Problem 8.5 (CMIMC 2018/7)**.** Compute the value of

$$\sum_{k=0}^{2017} \frac{5 + \cos\left(\frac{k\pi}{1009}\right)}{26 + 10\cos\left(\frac{k\pi}{1009}\right)}.$$

Hints: 13 5

Problem 8.6 (HMMT Guts 2014/31)**.** Evaluate

$$\sum_{k=1}^{1007} \left(\cos\left(\frac{k\pi}{1007}\right)\right)^{2014}.$$

Hints: 101 17

Problem 8.7. Consider a rectangle $ABCD$ such that side AB has length n and side BC has length m. A circle is drawn with center E at the midpoint of side BC such that it is tangent to the diagonal AC. Determine the radius of this circle in terms of n and m. **Hints**: 73 106

Problem 8.8. Find the number of intersections of the parabola $x^2 = 2p(y + \frac{p}{2})$ and the line $x\cos\theta + y\sin\theta = p\sin\theta$. **Hints**: 30 50 3

Problem 8.9. For $a \neq b$,

$$a^2\sin\theta + a\cos\theta - 1 = 0,$$
$$b^2\sin\theta + b\cos\theta - 1 = 0.$$

Let l be the line determined by (a, a^2) and (b, b^2). Find the number of intersections of l and the unit circle. **Hints**: 14 16 25

Problem 8.10. Let ABC be a triangle with inradius r and circumradius R. Show that

1. $4\sin A\sin B\sin C = \sin 2A + \sin 2B + \sin 2C$.
2. if $\sin^2 A + \sin^2 B + \sin^2 C = 2$ then ABC is a right triangle.
3. if ABC is a cute then $2\cos A\cos B\cos C + \cos 2A + \cos 2B + \cos 2C = -1$.
4. $[ABC] = 2R^2\sin A\sin B\sin C$.
5. $a\cos A + b\cos B + c\cos C = \frac{abc}{2R}$.
6. $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.
7. $a\cos B + b\cos C + c\cos A = \frac{a+b+c}{2}$.

Hints: 29

Problem 8.11. Find the dihedral angle between adjacent faces of a:

1. regular tetrahedron,
2. regular octahedron,
3. regular dodecahedron, and
4. regular icosahedron.

Hints: 45

§A Appendix A: List of Theorems and Definitions

List of Theorems

List of Definitions

§B Appendix B: Hints

1. Without vectors, simply calculate the lengths of the sides of triangle PQO, where O is the origin. This is easily done by 3D Pythagorean Theorem. With vectors, calculate the normal vector.
2. Try to factor the given expression. Maybe the form at the end will help you a bit?
3. We get $y = \frac{p}{1 \pm \cos\theta}$. Isn't that cool?
4. Find $\tan\frac{\pi}{4}$. What does this have to do with what we want?
5. The value from the last hint is $\theta = \frac{\pi k}{1009}$. Now try to complete the square on part of the denominator (exclude $\sin^2\theta$). If you would like the full solution, look here.
6. $\frac{1}{2}ab\sin C$ works for triangles - does a similar thing work for quadrilaterals (using the diagonals)?
7. We either have $\tan x = -1$, or $\cos x = 0$. Why?
8. Your answer will come from sector $-$ triangle $-$ triangle. But can you find special properties of these triangles?
9. Try to find $30 - 60 - 90$ triangles.

10. This looks suspiciously like Pythagorean Theorem, and this is in a trigonometry handout. What could this mean?
11. Most of the work is using power of a point. Can you find two points that have the same power? Can you then find $AB > AC$ and AC?
12. Can you evaluate one of them very easily? Also, try to rewrite such that all the cos terms are less than $\frac{\pi}{2}$.
13. 26 isn't a nice number. What about $25 + \cos^2\theta + \sin^2\theta$? What θ will make this good?
14. Subtraction seems nice.
15. This is a rather direct application of complex numbers. I'll leave you to it, in order to not spoil anything.
16. What is the slope of the line?
17. Note that $\cos\left(\frac{\pi k}{1007}\right) = \frac{1}{2}(\omega^k + \omega^{-k})$. Now bash with sums. The full solution is located here.
18. Try to graph it and use the Bounds of $\sin\theta$ and $\cos\theta$ (specifically $\sin\theta$).
19. Consider the cases $x > 1, x < 1$, and $x = 1$ all separately. Try to find patterns in the case $x > 1$.
20. BD is a good candidate for the Law of Cosines.
21. Hey! You know a lot about the properties of sine! Simplify your expression into something manageable. Then compute using basic trigonometric values.
22. Note that if $a^3 + b^3 = (a+b)^3$, this rearranges to $3ab(a+b) = 0$. What can you take a and b as? What can you conclude? You should have 3 cases - just solve all of them!
23. For the first three, substitute them into the Addition-Subtraction Identities, with $\alpha = \beta$. The last three immediately follow from their definitions as reciprocals.
24. Try to substitute $x = i$ instead.
25. The slope was $a + b$, and we got that from the first hint as $-\tan\theta$. Now intersect it with $x^2 + y^2 = 1$.
26. Using the above diagram, we can see that $\cos(\alpha+\beta) = EB = CB - CE = CB - AF$.
27. Have you considered complex yet? Roots of Unity!
28. Try using the exponential form (e^x). If you want a full proof, look here.
29. All I'll say is refer to Trigonometric Substitution, and good luck.
30. Find x in terms of y, p, and θ, then substitute into the first equation.
31. Apply the Magnitude of Cross Product theorem.
32. This looks like a partial decomposition problem - there is sin in the denominator so which trig function do you think of? There are two possible answers.
33.
34. Find $\cos 2x$. What does this tell us about $2x$?
35. Consider the first two and last two terms separately. Use the Sum to Product Identities on each of them.
36. As said before, don't wait for some magic. Heron's formula does the trick.
37. Try to assume $z = a = bi$ and cross multiply. Separate the real and imaginary parts as well.
38. Try using Bretschneider's Formula. For those that have given up, I believe the full proof is in here. You can also search up "the area of a tangential quadrilateral".
39. Even though things don't work out as you imagine, use the facts in Potpourri to get a lot of cancellation.
40. Try to simply factor and reduce as much as possible.
41. There's a lot of symmetry going on, and 101 seems like a random number. What if we tried a simpler case, where 101 was replaced with 2 or 3? The result follows pretty fast from Engineer's Induction. See if you can prove it for all n, however.

42. You may need to use Pitot's Theorem.
43. The shoelace formula is one of the best ways to find the area.
44. Applying a few basic trigonometric identities will get you far in this question. Don't think to hard about it, it's just relating sin and cos.
45. Just bash with the Dihedral Angle Formula.
46. Try to divide by $\cos^2\theta$. It would help, especially to get an equation all in terms of $\tan\theta$ (use section 3 of my Polynomials in the AIME Handout to finish the problem).
47. The $\cos^3$ is annoying - try to start with Sum to Product Identities on $\cos 3x + \cos 5x$. What is the result? How does it relate to what you have in the problem.
48. It's begging for trigonometry, but it's not very obvious how to calculate this. Try to assign arc lengths.
49. Try to use our bounds on $\sin\theta$ and $\cos\theta$ instead of rederiving them.
50. Remember that p is a constant. Find y using the quadratic formula.
51. Try to let $\tan\angle ABC = x$. What can you get?
52. You probably know $\angle A$. Law of Cosines should finish it off.
53. Choosing a nice point can give you the answer, but not the proof. Perhaps trigonometric bashing will help - if a nicer proof is what you desire, go to the next hint. If you are okay with a little bash, try Ratio Lemma.
54. Just use the Double Angle Identities with $\alpha = \frac{1}{2}\theta$.
55. Find a cubic in terms of $\sin 18°$. Can you find an easy root (see my Polynomials in the AIME Handout and section 2)? Note that this root is not $\sin 18°$.
56. A non-trigonometric way to attempt this problem is to realize the pattern for equilateral triangles, squares, regular pentagons, etc. With this knowledge, it may help in a trigonometric proof.
57. Use the formula (Volume of a Parallelepiped) given above.
58. Try to write down as many Law of Sines equations as possible. Can you combine some such that (even though it looks very contrived) the only term is $\angle DBA$.
59. I suggest Polar Complex Numbers.
60. It helps if you know Newtons' Sums (see my Polynomials in the AIME Handout and section 4). Otherwise, use the same strategy as in the above problem, by finding $\cos^2 x \sin^2 x$.
61. The identity $x^2+y^2 = (x+y)^2 - 2xy$ comes in useful, with $x = \sin t$ and $y = \cos t$. You can get a system of equations, and try to solve it.
62. Similar triangles help, and try to use $\tan\angle BEF$.
63. Try to expand and factor. Remember 107 is prime!
64. Use the Law of Cosines and $\frac{1}{2}ab\sin C$ to get $[ABCD]$ in terms of $a, b, c, d, \angle A, \angle C$.
65. For both of the trig functions, write them with a common denominator. Which one seems the easiest to use (maybe Addition-Subtraction Identities can help)?
66. Use the definition to derive the general formula, then plug in $\frac{\pi}{6}$.
67. Use the Law of Sines (extended version) to find AC and BD of any quadrilateral. Can you finish off with the area (you can guess - it's most likely symmetric).
68. Try to let the angle between AB and the x-axis be θ. Use trig functions to find the value of θ and the rest of the coordinates.
69. Consider the graph below, as well as the domain of a tangent graph.
70. Use that Double Angle Identities to write $\sin 2\theta$ is in form of $\sin\theta$ and $\cos\theta$. Does this look familiar?

71. What section is this again? That's right use complex numbers! Specifically, $\omega = e^{\frac{2\pi i}{90}}$. The solution is available here.

72. Like the last example, look at the unit circle below.
73. Let the circle be tangent to AC at F. What is EF in terms of EC and $\angle ACB$?
74. I suggest Polar Complex Numbers.
75. Look back at the definitions.
76. There is obviously a pattern between a, b, and c. What happens when we add up the angles?
77. Remember, sometimes the substitution $y = x + \frac{1}{x}$ helps.
78. I suggest to let $z_k = x_k + (mx_k + b)i$ where the equation of line L is $y = mx + b$. Now solve for m.
79. Use $w = a + bi$ and $z = c + di$ and use the definition.
80. You could fool around with the Law of Sines, but it seems easy enough to just drop perpendiculars.
81. Try to draw in the feet of the perpendiculars as well as the center to the intersection point.
82. Try to factor $P(x)$. Maybe find a few roots?
83. This was added just to make sure the reader understood vectors. Well, do you? Think of how adding vectors works.

84. Try to use the Law of Cosines on $\triangle ADC$ and $\triangle ABC$. Find the length of the median.
85. Don't look for some smart trig identity - Heron's formula.
86. Don't use the diagrams - use the definition of Tangent. Once you get a nasty expression in terms of $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$, try to divide both the numerator and denominator by $\cos\alpha\cos\beta$.
87. Maybe you can work with something like $2\angle DBA$? Write a cubic in $\cos 2\angle DBA$.
88. You'll get a quadratic - which you can hopefully solve for $\sin x + \cos x$ and then solve for $\sin x \cos x$. Can you find $\sin 2x$ from the Double Angle Identities? Can this help you find what x is?
89. There are 6 different cases you should have - remember that the Law of Sines (or alernatively drawing radii) will make your life much easier.
90. Refer to AIME 1995/7 to see the same method used to solve this problem.
91. It typically helps to draw a diagram. Draw one! Can you make inferences about y-coordinates?
92. Look at the unit circles below for a bit of intuition.
93. It's on similar lines to the previous problem - use Double Angle Identities and Addition-Subtraction Identities to break down the problem into $\sin x, \cos x$.
94. Try relating $36°$ and $54°$ by some of the identities in the Potpourri.
95. Just because the question said to bound these functions does not mean they have a bound. Think about the graphs of the functions.
96. There's a fancy construction solution - but use the law of Sines should get you pretty far. Alternatively, Trig Ceva could work.
97. Try using the Double Angle Identities by multiplying by $\sin\frac{\pi}{9}$.
98. Rearrange the position of the chords to form a triangle. Then, use trigonometry!
99. Sometimes expanding the sum/difference of angles helps a lot.
100. Like the last example, use the definition to derive the general formula, then plug in $225°$.
101. Try roots of unity, with $\omega = e^{\frac{2\pi i}{2014}}$.
102. Try to see if a direct application of the roots of unity helps here.

103. Seems so perfect for the Law of Sines. Find $\angle B$ and $\angle C$.
104. Find $\sin \frac{\pi}{6}$. What does this have to do with what we want?
105. To finish, just factor the cubic! It should be a familiar angle.
106. Now calculate $\angle ABC$ in terms of m and n. Use $\triangle ABC$.