Larger empty axis-parallel boxes

Dylan Yu   Alec Sun

Abstract

We show that, for every set of n points in a 2-dimensional unit square, there is an empty axis-parallel box of volume at least $\frac{1.5059}{n}$ as $n \to \infty$. This improves upon the previous best lower bound of $\frac{1.5047}{n}$ as $n \to \infty$.

1 Introduction

In this paper, we find, for every set P of n points in a 2-dimensional unit square, a lower bound for the area of the largest axis-parallel box in a unit square that does not contain any of the n points.

Definition 1 (Box)**.** *A* box *is a Cartesian product of open intervals. Given a set P, We say that a box is* empty *if $B \cap P = \emptyset$.*

For the 2-dimensional case (i.e., a rectangle), we take two Cartesian products.

Let $m(P)$ (called the *dispersion* of P) be the volume of the largest empty box contained in $[0,1]^2$. Let $m_2(n)$ be the largest number such that every n-point set $P \subset [0,1]^2$ admits an empty box of volume at least $m_2(n)$. Alternatively, $m_2(n) = \min m(P)$, where the minimum is taken over all sets P.

Several works found $m_2(n)$ to be at least $\frac{1}{n+1}$, including [4, 8, 14]. The first non-trivial bound of $m_2(n) \geq \frac{5}{4(n+5)}$ was observed by Dumitrescu and Jiang [9]. Aistleitner, Hinrichs, and Rudolf [2] then bounded $c_2 \geq \frac{1}{4}$ for

$$c_2 := \lim_{n\to\infty} n m_2(n),$$

and Bukh and Chao [5] then improved upon this to $c_2 \geq 1.5047$, the proof of which is reproduced in Appendix B. Note that the upper bound of $c_2 \leq 1.8945$ was achieved by Kritzinger and Wiart [11]. Other algorithms focusing on the planar problem have been proposed over the years, including [1, 3, 6, 7, 13].

The motivation for estimating $m_2(n)$ (and consequently c_2) comes from several independent subjects. Rote and Tichy [14] were motivated by the relations to ε-nets in discrete geometry and the relations to discrepancy theory. The dispersion also arose in the problem of estimating rank-one tensors [4, 10, 12] and in Marcinkiewicz-type discretizations [16]. Rank-one tensors in particular yield real-world applications in machine learning, engineering, and areas of physics (e.g., fluid mechanics, electrodynamics, and general relativity). Shamos' work on the empty circle problem is motivated by the facilities location problem, but one in which a new facility should be positioned as far away from other facilities as possible (e.g. to avoid polluting the nearby area or to avoid competition) [15]. This same use-case is applicable to the empty rectangle problem addressed in this paper.

2 Methodology

We adopt the notation of Bukh and Chao [5] (for 2-dimensional boxes) as below.

Let P denote an n-point set in $[0,1]^2$ and denote by $P - t$ the set of points shifted by $t \in \mathbb{R}^2$. Let $\delta > 0$ be a parameter to be chosen later. We use the same strategy as in [5] of considering a random shift $t \in [\delta, 1-\delta]^2$ as a reference point which is the center of a shifted box $B = [-\delta, \delta]^2$. To find an empty box with large area, we shave a side off of B for every point in $P' = (P - t) \cap B$. For each point $p \in P'$ there is a coordinate of maximum absolute value, which we call *dominant* for p. We will shave off the dominant coordinate for p, since the shaving off the largest coordinate generally results in shaving off the least area. Writing the coordinates of p as $p = (p_1, p_2)$, define

$$a_i := \min\left\{-p_i : i \text{ is dominant for } p \in P' \text{ and } p_i \leq 0\right\},$$
$$b_i := \min\left\{+p_i : i \text{ is dominant for } p \in P' \text{ and } p_i \leq 0\right\}.$$

If the set in the definition of a_i or b_i is empty, then we set $a_i = \delta$ or $b_i = \delta$. Note that the box $B' := (-a_1, b_1) \times (-a_2, b_2)$ is precisely the box that results when we shave off the dominant coordinate for each point $p \in P'$. Hence B' is disjoint from $P - t$ and is contained in B. We can assume that $a_i, b_i > 0$, since if $a_i = 0$ or $b_i = 0$ it must be the case that $P - t$ contains the origin, equivalently $t \in P$. Such shifts t have density 0 in the set of possible shifts $t \in [\delta, 1-\delta]^2$ and hence can be ignored.

A key lemma in [5] (Lemma 4 in the article) is the following (where box and volume correspond to rectangle and area in 2D space):

Lemma 2. *The volume of B' is at least* $\operatorname{vol}(B) \cdot \prod_{p \in P'} \sqrt{\frac{\|p\|_\infty}{\delta}}$.

The proof of Lemma 2 uses the AM-GM inequality to lower bound each side length $a_i + b_i$ of B' in terms of the product of the ℓ^∞-norms of the points in $P - t$ (see Appendix A). Our main observation is that often this inequality can be improved because for a typical shift t, one would not expect a_i and b_i to be exactly equal, hence we have $a_i + b_i > 2\sqrt{a_i b_i}$. How much multiplicative improvement we get over the AM-GM inequality depends on how far apart a_i and b_i are. Letting $\kappa_i = \max\left\{\frac{a_i}{b_i}, \frac{b_i}{a_i}\right\}$ be the *eccentricity parameter* for coordinate i, then we have

$$a_i + b_i \geq \frac{1+\kappa_i}{\sqrt{\kappa_i}} \sqrt{a_i b_i}.$$

Note that if $\kappa_i = 1$, then we recover the AM-GM inequality. We use this refined inequality to improve Lemma 2 and yield a stronger lower bound for c_2.

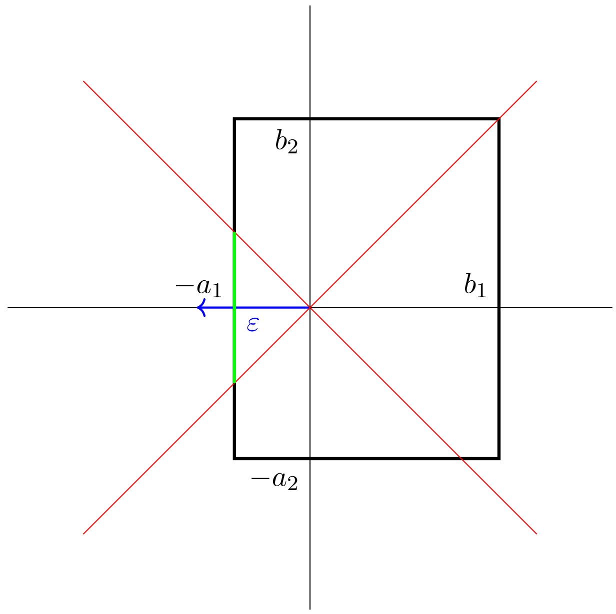
3 Optimizing B'

Let the average area of boxes B' over all choices of t be A. Since [5] proves the maximum area is at least $\frac{c_2}{n}$, with equality when all $\operatorname{vol} B' = \frac{c_2}{n}$, it follows that $A > \frac{c_2}{n}$. For each reference point t, consider t and its translates in each of the 4 cardinal directions by $\varepsilon = \frac{k}{\sqrt{n}}$ for $k = \frac{\sqrt{nA}}{3.5} > \frac{\sqrt{c_2}}{3.5} \approx 0.350$.

Definition 3 (Stable)**.** *Call a box* stable *if all four of t's translates lie within B', and call a box* unstable *otherwise.*

Lemma 4. *Define an* actual *point as one of the n points in set P. In an unstable box B', the reference point must be within $k\sqrt{\frac{2}{n}}$ of one of the n actual points.*

Proof. Without loss of generality, let $a_1 < \varepsilon$.



Clearly, the actual point forming the $x = -a_1$ side of B' must lie on $x = -a_1$, and it must be contained between $y = -x$ and $y = x$ or else coordinate 2 would be dominant. As such, the actual point can be at most $\varepsilon\sqrt{2} = k\sqrt{\frac{2}{n}}$ away from the reference point as desired. □

Lemma 5. *Either* $A > \frac{1.560}{n}$*, or the ratio of stable boxes over all boxes is greater than 0.2.*

Proof. Consider an unstable box B'. By Lemma 4, the region in which the reference point can lie must be within a circle of radius $k\sqrt{\frac{2}{n}}$ around at least one of the n actual points, which gives us an area of at most

$$\frac{2\pi k^2}{n} \cdot n = 2\pi k^2,$$

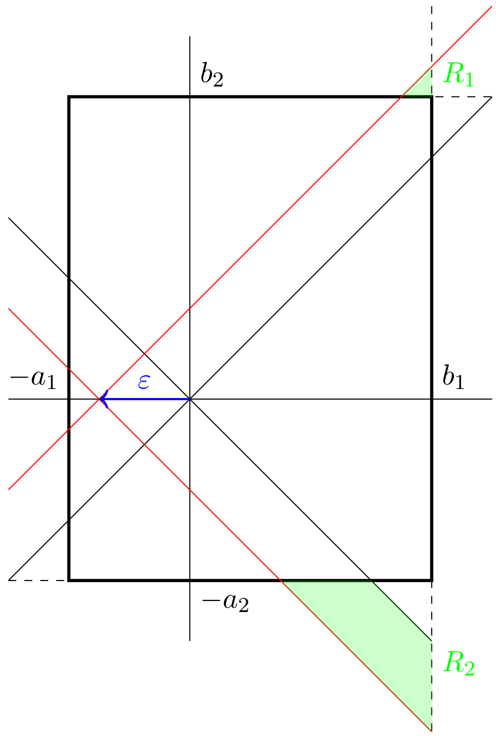
implying that the ratio of unstable boxes is less than $2\pi k^2$.

- If $2\pi k^2 > 0.8$, then $A > 0.8 \cdot \frac{3.5^2}{2\pi n} > \frac{1.559}{n}$, improving the lower bound on c_2 given in [5].
- If $2\pi k^2 < 0.8$, then $k < 0.357$ and the ratio of stable boxes is at least $1 - 2\pi k^2 > 0.2$.

Note that the former case results in an average area (and therefore maximum area) of at least $\frac{1.550}{n}$, which exceeds the improvement of $\frac{1.5059}{n}$ we aim to show in this paper. □

4 Stable Box Analysis

By Lemma 5, we have a substantial ratio (i.e., greater than 0.2) of stable boxes. Consider one of those boxes $B' = (-a_1, b_1) \times (-a_2, b_2)$. Without loss of generality, let $a_1 < a_2$ and $b_i = \kappa_i a_i$ for $i = 1, 2$.



Translate the origin from the reference point a distance ε in the $-a_1$ direction. Let a_1', b_1' denote the new a_1, b_1 when using the new origin. All equations presented below use the untranslated reference point as the origin.

Lemma 6. *The only set S of points that can affect a_1', b_1' are the ones existing in:*

- *the region R_1 between $y = x$ and $y = x + \varepsilon$ lying above $y = b_2$, or*
- *the region R_2 between $y = -x$ and $y = -x - \varepsilon$ lying below $y = -a_2$.*

Proof. Note that there cannot be any points within the box – otherwise, a_i, b_i would be smaller. Define C as the following groups of points:

- $(b_1, \infty) \times (b_2, \infty)$,
- $(-\infty, -a_1) \times (b_2, \infty)$,
- $(-\infty, -a_1) \times (-\infty, -a_2)$, or
- $(b_1, \infty) \times (-\infty, -a_2)$.

Points in C cannot affect the box, since the dominant coordinate of any points in C are larger than the current choice of a_i, b_i for any origin within the box. Points lying above or below both $y = x + \varepsilon$ and $y = -x - \varepsilon$ are dominant for coordinate 2 after the change in origin, so they will not affect a_1', b_1'. Points lying to the left or right both $y = x$ and $y = -x$, because they are dominant for coordinate 1 before the change in origin, so they cannot exist for a_i, b_i to exist. As such, only points not in C lying between $y = x$ and $y = x + \varepsilon$ or between $y = -x$ and $y = -x - \varepsilon$ can affect a_1', b_1', which yields the desired result. □

The point that yields the smallest b_1' is either the leftmost point in R_1, $(b_2-\varepsilon, b_2)$, or the leftmost point point in R_2, $(a_2-\varepsilon, -a_2)$. Since $a_2-\varepsilon < b_2-\varepsilon$, we have that the smallest value of b_1' is $a_2-\varepsilon+\varepsilon = a_2$, implying that

$$\kappa_1' = \frac{b_1'}{a_1'} = \frac{a_2}{a_1-\varepsilon} > 1.$$

Given the minimum and maximum bounds on $\operatorname{vol} B'$ attained in Theorems 1 and 2 found in [5], we must have that $\operatorname{vol} B'$ is asymptotic to $\frac{1}{n}$. Since ε is $O\left(\frac{1}{\sqrt{n}}\right)$, it follows that the two sides are also at least $O\left(\frac{1}{\sqrt{n}}\right)$, since otherwise ε would be larger than the sides for sufficiently large n. Furthermore, the two sides must be exactly $O\left(\frac{1}{\sqrt{n}}\right)$ for $\operatorname{vol} B'$ to be asymptotic to $\frac{1}{n}$.

Lemma 7. *Both sides of a stable box have sides at most* $\frac{\operatorname{vol} B'}{2\varepsilon}$.

Proof. Since $\operatorname{vol} B' = (a_1+b_1)(a_2+b_2)$ and both sides exceed 2ε, the desired follows. □

Note that $\frac{\operatorname{vol} B'}{2\varepsilon}$ is asymptotic to $\frac{1}{\sqrt{n}}$.

The side lengths are $a_i+b_i = (1+\kappa_i)a_i$ (i.e. a_i, b_i are also asymptotic to $\frac{1}{\sqrt{n}}$), so it follows that $a_i, b_i < c\varepsilon$ for $c = \frac{\operatorname{vol} B'}{2\varepsilon^2} > 1$, where $i=1,2$. Say $\operatorname{vol} B' = \frac{\beta}{n}$ for some constant β. Then $c = \frac{\beta}{2k^2}$, so

$$\kappa_1' = \frac{a_2}{a_1-\varepsilon} > \frac{a_1}{a_1-\varepsilon} > \frac{c}{c-1} = \frac{\beta}{\beta-2k^2}.$$

Since $\beta < c_2 \approx 1.505$ (otherwise the bound on $\operatorname{vol} B'$ can already be improved), we have that

$$\kappa_1' > \frac{c_2}{c_2-2k^2} > \frac{c_2}{c_2 - 2\left(\frac{\sqrt{c_2}}{3.5}\right)^2} \approx 1.195.$$

5 Improving $\operatorname{vol} B'$

Lemma 8. *Let a shift* $t_i \in [\delta, 1-\delta]^2$ *induce a box* $B_i' = (-a_1^{(i)}, b_1^{(i)}) \times (-a_2^{(i)}, b_2^{(i)})$, *where* t_0 *describes some initial reference point and* t_i *for* $i=1,2,3,4$ *describes the four translates of that reference point by* ε *in the four cardinal directions. Define*

$$\kappa_1^{(i)} = \max\left\{\frac{a_1^{(i)}}{b_1^{(i)}}, \frac{b_1^{(i)}}{a_1^{(i)}}\right\}, \qquad \kappa_2^{(i)} = \max\left\{\frac{a_2^{(i)}}{b_2^{(i)}}, \frac{b_2^{(i)}}{a_2^{(i)}}\right\}$$

to be the eccentricity parameters for coordinates 1 and 2, respectively, where $i=0,\ldots,4$, *and define*

$$\kappa = \frac{1}{5}\sum_{i=0}^{4} \ln\left(\frac{1+\kappa_1^{(i)}}{2\sqrt{\kappa_1^{(i)}}}\right)\left(\frac{1+\kappa_2^{(i)}}{2\sqrt{\kappa_2^{(i)}}}\right)$$

as the total *eccentricity. Then*

$$\operatorname{vol}(B') \geq \operatorname{vol}(B) \cdot e^{\kappa} \cdot \prod_{p\in P'} \sqrt{\frac{\|p\|_\infty}{\delta}}.$$

Proof. The proof of Lemma 8 is analogous to that of Lemma 2:

$$\begin{aligned}
\operatorname{vol} B_i' &= (a_1^{(i)} + b_1^{(i)})(a_2^{(i)} + b_2^{(i)}) \\
&\geq \frac{1+\kappa_1^{(i)}}{2\sqrt{\kappa_1^{(i)}}} \cdot \frac{1+\kappa_2^{(i)}}{2\sqrt{\kappa_2^{(i)}}} \sqrt{(a_1^{(i)} + b_1^{(i)})(a_2^{(i)} + b_2^{(i)})} \\
&\geq \frac{1+\kappa_1^{(i)}}{2\sqrt{\kappa_1^{(i)}}} \cdot \frac{1+\kappa_2^{(i)}}{2\sqrt{\kappa_2^{(i)}}} \cdot (2\delta)^2 \prod_{p \in P'} \sqrt{\frac{\|p\|_\infty}{\delta}},
\end{aligned} \tag{1}$$

so we have

$$\begin{aligned}
\max_i \operatorname{vol} B_i' &\geq \frac{\sum_{i=0}^4 \operatorname{vol} B_i'}{5} \\
&\geq \sqrt[5]{\prod_{i=0}^4 \operatorname{vol} B_i'} \\
&\geq e^\kappa \cdot (2\delta)^2 \prod_{p \in P'} \sqrt{\frac{\|p\|_\infty}{\delta}}
\end{aligned} \tag{2}$$

as desired. □

Note that the total eccentricity $\kappa = \kappa(t)$ depends on the shift t, where we set $\kappa(t) = 0$ for $t \notin [\delta, 1-\delta]^2$.

We set the same weight functions F as in Theorem 1 and Theorem 3 of [5]. As described in the proof of Theorem 1 and Theorem 3 of [5], by Lemma 2 the problem of maximizing $\operatorname{vol}(B')$ as a function of $t \in [\delta, 1-\delta]^2$ is equivalent to minimizing $\frac{1}{4}\sum_{p \in P-t} F(p)$. Using our notion of eccentricity in Lemma 8, in order to maximize $\operatorname{vol}(B')$ we want to instead minimize the function

$$-\kappa(t) + \frac{1}{4} \sum_{p \in P-t} F(p).$$

Since $\kappa(t) \geq 0$ for all t, and for a positive density of t we have that $\kappa(t)$ is uniformly bounded away from 0, we can guarantee the existence of $t \in [\delta, 1-\delta]^2$ such that B' has a larger volume than was guaranteed in [5].

We now present the details of this argument. We integrate the function

$$-\kappa(t) + \frac{1}{4} \sum_{p \in P-t} F(p)$$

we want to minimize over all $t \in \mathbb{R}^2$ and compute

$$\begin{aligned}
\int_{t \in \mathbb{R}^2} \left(-\kappa(t) + \frac{1}{4} \sum_{p \in P-t} F(p) \right) dt &= -\int_{t \in \mathbb{R}^2} \kappa(t)\, dt + \frac{1}{4} \int_{t \in \mathbb{R}^2} \sum_{p \in P-t} F(p)\, dt \\
&= -\int_{t \in [\delta, 1-\delta]^2} \kappa(t)\, dt + \frac{1}{4} \int_{t \in \mathbb{R}^2} \sum_{p \in P-t} F(p)\, dt \\
&= -\int_{t \in [\delta, 1-\delta]^2} \kappa(t)\, dt + \frac{1}{4} \sum_{p \in P} \int_{x \in \mathbb{R}^2} F(x)\, dx \\
&= -\int_{t \in [\delta, 1-\delta]^2} \kappa(t)\, dt + \frac{M}{4},
\end{aligned} \tag{3}$$

where $M := n\int_{x\in B} F(x)\,dx$. Observe that the term $-\int_{t\in[\delta,1-\delta]^2}\kappa(t)\,dt$ in Eq. (3) represents the multiplicative improvement on $\mathrm{vol}(B')$ we achieve over [5]. By the Pigeonhole Principle, there exists $t\in[\delta,1-\delta]^2$ such that

$$-\kappa(t)+\frac{1}{4}\sum_{p\in P-t}F(p)\le -\frac{\int_{t\in[\delta,1-\delta]^2}\kappa(t)\,dt}{(1-2\delta)^2}+\frac{M}{4(1-2\delta)^2}.$$

Defining

$$\kappa_{\mathrm{avg}}=\frac{\int_{t\in[\delta,1-\delta]^2}\kappa(t)\,dt}{(1-2\delta)^2},$$

we conclude by Lemma 8 that for this choice of t we have

$$\begin{aligned}\mathrm{vol}(B')&\ge \mathrm{vol}(B)\cdot\exp\left(\kappa(t)-\frac{1}{4}\sum_{p\in P'}F(p)\right)\\&\ge \mathrm{vol}(B)\cdot\exp\left(\kappa_{\mathrm{avg}}-\frac{M}{4(1-2\delta)^2}\right)\\&\ge e^{\kappa_{\mathrm{avg}}}\cdot\frac{R_0}{n}\exp\left(\frac{M}{4(1-2\delta)^2}\right).\end{aligned}$$

By the proof of Theorem 3 in [5], we may take $n\to\infty$ to find

$$c_2\ge e^{\kappa_{\mathrm{avg}}}\cdot\left(\frac{4}{e}(1+e^{-4})\right).$$

At least one of the $\kappa_i^{(j)}$ terms (for $i=1,2$ and $j=0,\ldots,4$) is greater than 1.195 – without loss of generality, let it be $\kappa_1^{(1)}$. Then

$$\kappa\ge\frac{1}{5}\ln\left(\frac{1+\kappa_1^{(1)}}{2\sqrt{\kappa_1^{(1)}}}\right)\approx 0.0007932,$$

with equality when all other terms are 1 (the minimum). Taking a constant function $\kappa=\frac{1}{5}\ln\left(\frac{1+\kappa_1^{(1)}}{2\sqrt{\kappa_1^{(1)}}}\right)$, it follows that

$$\kappa_{\mathrm{avg}}\ge\frac{(1-2\delta)^2\kappa}{(1-2\delta)^2}\ge 0.0007932,$$

improving c_2 to $e^{0.0007932}\cdot 1.5047\approx 1.5059$.

References

[1] A. Aggarwal and S. Suri. "Fast Algorithms for Computing the Largest Empty Rectangle". In: *Proceedings of the Third Annual Symposium on Computational Geometry*. SCG '87. Waterloo, Ontario, Canada: Association for Computing Machinery, 1987, pp. 278–290. ISBN: 0897912314. DOI: 10.1145/41958.41988. URL: https://doi.org/10.1145/41958.41988.

[2] Christoph Aistleitner, Aicke Hinrichs, and Daniel Rudolf. *On the size of the largest empty box amidst a point set*. 2015. DOI: 10.48550/ARXIV.1507.02067. URL: https://arxiv.org/abs/1507.02067.

[3] Mikhail J Atallah and Greg N Frederickson. "A note on finding a maximum empty rectangle". In: *Discrete Applied Mathematics* 13.1 (1986), pp. 87–91. ISSN: 0166-218X. DOI: https://doi.org/10.1016/0166-218X(86)90071-5. URL: https://www.sciencedirect.com/science/article/pii/0166218X86900715.

[4] Markus Bachmayr et al. *Approximation of high-dimensional rank one tensors*. URL: https://www.math.tamu.edu/~rdevore/publications/155.pdf.

[5] Boris Bukh and Ting-Wei Chao. *Empty axis-parallel boxes*. 2020. DOI: 10.48550/ARXIV.2009.05820. URL: https://arxiv.org/abs/2009.05820.

[6] B. Chazelle, R. L. Drysdale, and D. T. Lee. "Computing the Largest Empty Rectangle". In: *SIAM Journal on Computing* 15.1 (1986), pp. 300–315. DOI: 10.1137/0215022. eprint: https://doi.org/10.1137/0215022. URL: https://doi.org/10.1137/0215022.

[7] Amitava Datta. "Efficient algorithms for the largest rectangle problem". In: *Information Sciences* 64.1 (1992), pp. 121–141. ISSN: 0020-0255. DOI: https://doi.org/10.1016/0020-0255(92)90115-O. URL: https://www.sciencedirect.com/science/article/pii/002002559290115O.

[8] Adrian Dumitrescu and Minghui Jiang. *Computational geometry column 60*. Dec. 2014. URL: https://dl.acm.org/doi/10.1145/2696081.2696096.

[9] Adrian Dumitrescu and Minghui Jiang. *On the largest empty axis-parallel box amidst n points*. 2009. DOI: 10.48550/ARXIV.0909.3127. URL: https://arxiv.org/abs/0909.3127.

[10] David Krieg and Daniel Rudolf. *Recovery algorithms for high-dimensional rank one tensors*. 2017. DOI: 10.48550/ARXIV.1711.03986. URL: https://arxiv.org/abs/1711.03986.

[11] Ralph Kritzinger and Jaspar Wiart. "Improved dispersion bounds for modified Fibonacci lattices". In: (2020). DOI: 10.48550/ARXIV.2007.02297. URL: https://arxiv.org/abs/2007.02297.

[12] Erich Novak and Daniel Rudolf. *Tractability of the approximation of high-dimensional rank one tensors*. 2014. DOI: 10.48550/ARXIV.1402.5011. URL: https://arxiv.org/abs/1402.5011.

[13] Marian Orlowski. "A new algorithm for the largest empty rectangle problem". In: *Algorithmica* 5 (2005), pp. 65–73.

[14] G. Rote and R. F. Tichy. *Quasi-monte-carlo methods and the dispersion of point sequences*. Apr. 1996. URL: https://dl.acm.org/doi/10.1016/0895-7177%2896%2900036-2.

[15] Michael Ian Shamos. *Computational Geometry*. May 1978. URL: http://euro.ecom.cmu.edu/people/faculty/mshamos/1978ShamosThesis.pdf.

[16] V. N. Temlyakov. *Universal discretization*. 2017. DOI: 10.48550/ARXIV.1708.08544. URL: https://arxiv.org/abs/1708.08544.

A Proof of Lemma 2

We repeat the proof from [5].

Fix any coordinate $i = 1, 2$.

- Suppose first that the two sets in the definitions of a_i, b_i are non-empty. Then if $p, q \in P - t$ are points such that $a_i = -p_i, b_i = q_i$, we have by the AM-GM inequality that

$$\frac{a_i + b_i}{2\delta} \geq \frac{\sqrt{a_i b_i}}{\delta} = \sqrt{\frac{\|p\|_\infty}{\delta}} \cdot \sqrt{\frac{\|q\|_\infty}{\delta}}. \tag{4}$$

- Suppose next that only one of the two sets are non-empty. Without loss of generality, let $a_i = -p_i$ for some $p \in P - t$ and $b_i = \delta$ (the other case is symmetric). Then by a similar application of the AM-GM inequality, we obtain

$$\frac{a_i + b_i}{2\delta} \geq \sqrt{\frac{\|p\|_\infty}{\delta}}. \tag{5}$$

Taking the product of Eq. (4) and Eq. (5) as appropriate over $i = 1, 2$, and noting that every point has only one dominant coordinate, we have

$$\operatorname{vol} B' = (2\delta)^d \cdot \prod_{i=1}^{d} \frac{a_i + b_i}{2\delta} \geq (2\delta)^d \prod_{p \in P'} \sqrt{\frac{\|p\|_\infty}{\delta}}.$$

B Proof of $c_2 \geq 1.5047$

We repeat the proof from [5].

Let $R_0 > 0$ be a parameter such that $R_0 \leq n$, and let $\delta = \frac{1}{2}\sqrt{\frac{R_0}{n}}$. Let $f \colon [0, R_0] \to \mathbb{R}^+$ be some weight function. We adopt the convention that $f(R) = 0$ if $R \geq R_0$.

Let B be a square of area $\frac{R_0}{n}$ centered at the origin, i.e., $B := [-\delta, \delta]^2$. Using f, we define a function on $\mathbb{R}^2$ by $F(x) := f(4r^2 n)$ for $\|x\|_\infty = r$. Because f vanishes outside $[0, R_0]$, the function F vanishes outside B. Define $M := n \int_B F(x)\, dx$, and note that

$$M = \int_{r=0}^{\delta} \left(2^d n r^{d-1} d \cdot f(2^d r^d n)\right) dr = \int_0^{R_0} f(R)\, dR.$$

Because

$$\int_{t \in \mathbb{R}^2} \sum_{p \in P - t} F(p)\, dt = \sum_{p \in P} \int_{t \in \mathbb{R}^2} F(p - t)\, dt = \sum_{p \in P} \int_{x \in \mathbb{R}^2} F(x)\, dx = M,$$

it follows by the Pigeonhole Principle that there exists $t \in [\delta, 1 - \delta]^2$ such that

$$\sum_{p \in P - t} F(p) \leq \frac{M}{(1 - 2\delta)^2},$$

for otherwise $\int_{[\delta, 1-\delta]^2} \sum_{p \in P - t} F(p)\, dt > M$. It suffices to find a large box inside B that is empty with respect to the set $P' := (P - t) \cap B$, for then we may obtain an empty box of the same volume inside $[0, 1]^2$ after translating by t – Lemma 2 is exactly this.

B.1 Simple weight function (proof of Theorem 1 in [5])

We choose $R_0 = 2d$ and $f(R) = \ln \frac{R_0}{R}$. The condition $R_0 \le n$ is satisfied unless $2d \le n$, but in that case Theorem 1 in [5] holds vacuously.

With this choice of R_0 and f, we obtain $M = \int_0^{R_0} f(R)\, dR = R_0$ and $F(x) = d \log \frac{\delta}{\|x\|_\infty}$ on B. By Lemma 2, we have

$$\begin{aligned}
\operatorname{vol} B' &\ge \frac{R_0}{n} \exp\Big(-\frac{1}{2} \sum_{p \in P'} \log \frac{\delta}{\|p\|_\infty}\Big) \\
&= \frac{R_0}{n} \exp\Big(-\frac{1}{4} \sum_{p \in P'} F(p)\Big) \\
&\ge \frac{R_0}{n} \exp\left(-\frac{M}{4(1-2\delta)^2}\right) \\
&= \frac{R_0}{n} \exp\left(-\frac{1}{\left(1-\sqrt{\frac{R_0}{n}}\right)^2}\right) \\
&\ge \frac{R_0}{n} \exp\left(-\frac{1}{\left(1-\frac{2}{\sqrt{n}}\right)^2}\right) \\
&\ge \frac{R_0}{n} \exp\left(-\frac{1}{1-\frac{4}{\sqrt{n}}}\right) \\
&\ge \frac{1}{n} \cdot \frac{4}{e} \left(1 - \frac{8}{\sqrt{n}}\right),
\end{aligned}$$

where the last two lines come from $\exp\left(-\frac{1}{1-x}\right) \ge e^{-1}(1-2x)$.

B.2 Better weight function (proof of Theorem 3 in [5])

By the same token, we may prove Theorem 3 by taking $R_0 = 3.695, T = 0.1016$, and

$$f(R) := \begin{cases} \ln \frac{R_0}{T} & R \le T \\ \ln \frac{R_0}{R} & T < R \le R_0 \end{cases}.$$

With these choices, we obtain $M = R_0 - T$ and

$$\operatorname{vol} B' \ge \frac{R_0}{n} \exp\left(-\frac{M}{4(1-2\delta)^2}\right),$$

and taking the limit $n \to \infty$, the bound on c_2 follows.